MECHANICS OF PARTICLES AND RIGID BODIES

By the same Author

APPLIED ELASTICITY
with diagrams

MECHANICS OF PARTICLES AND RIGID BODIES

BY

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WITH DIAGRAMS

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PREFACE TO THIRD EDITION

In this edition a few changes have been made which it is hoped are improvements of method or argument. These are not very extensive, and they have been worked in so as to alter the rest of the text as little as possible. Among these changes the most important are Arts. 270 to 272 on the rates of change of a vector and the application of this to velocities and accelerations. This method has the advantage of bringing under one set of rules a number of processes which appeared all different in the earlier editions. Another change, closely connected with the one just mentioned, is the representation in Chapter XXIII of the momentum of a rigid body by means of a vector acting in a definite line, together with a momentum-couple. The introduction of the momentum-couple enables us to apply the same operations to momentum as we had already applied to forces in Chapter III.

The rest of the changes are very small alterations or additions.

JOHN PRESCOTT.

June, 1929.

FROM THE PREFACE TO THE SECOND EDITION

THE object of the present book is to supply a text-book on statics and dynamics suitable for students taking these subjects up to the standard of a pass degree at a British university. The author's intercourse throughout several years with students and teachers of engineering in a technological college has resulted in giving a utilitarian bias to the book; but this, I hope, will be accounted a virtue, for mechanics is surely a utilitarian subject.

An attempt has been made to supply the answer to every question set at the ends of the chapters when the answer is not implied in the form of the question. For the labour of verifying most of these answers and correcting many of them, I am greatly indebted to Mr. Louis Toft, M.Sc.

This edition is a reprint of the first edition with just a few small corrections, and the alteration in two questions of the given speeds of aeroplanes to something like the speeds attained at the present day.

JOHN PRESCOTT.

January, 1923

CONTENTS

PART I

STATICS

PART II

DYNAMICS OF A PARTICLE

PART III

DYNAMICS OF A RIGID BODY

MECHANICS OF PARTICLES AND RIGID BODIES

CHAPTER I

PRELIMINARY

1. THE subject of mechanics is motion and the causes of motion. Sometimes we are concerned with motion alone, and sometimes with the causes alone. The cause of motion is force. Mechanics deals also with forces which mutually annul each other when they act on a rigid body. That part of the subject which deals with forces which mutually balance, and therefore cause no motion, is called *Statics*. The part which deals with forces causing motion is called *Dynamics*. The part which deals with motion alone and takes no account of its cause is regarded as a branch of dynamics, and is called *Kinematics*.

At the beginning of the subject it is necessary to give a few definitions of terms which are in common use, but to which precise meanings must now be attached.

2. **Definitions.**

Matter is something we shall not try to define.

A *Body* is a portion of matter bounded by a closed surface.

The *Mass* of a body is the quantity of matter composing the body.

It will be shown later how the masses of bodies of different substances are compared. It is easy to understand what is meant by equal masses of the same substance, but without some convention in addition to the preceding definitions there is no means of determining what is meant by equal masses of different substances. It is shown in Art. 47 that Newton's laws of motion (which are the postulates of mechanics) imply a definition of equal masses.

A *Particle* is a body of infinitely small dimensions; or, a mathematical point endowed with mass. The particle, as defined here, does not really exist; for there can be only an infinitely small mass in an infinitely small space. But it is a convenient mathematical fiction which simplifies mechanical reasoning. Any finite body is regarded as composed of particles with infinitely small masses.

Displacement.—If a particle is, at one instant, at a point A, and, at a later instant, at a point B, its displacement in the interval is the distance AB in the direction from A to B.

The *Velocity* of a particle is the rate of increase of its displacement. Since velocity is displacement in unit time, it has, like displacement, a direction as well as magnitude.

Force.—That which changes the motion of a body from uniform motion in a straight line, or which sets a body in motion, is called a force.

There are forces which do not seem to be embraced in the preceding definition. The physical action which we call a force may exist, but no motion may ensue, because its effect is exactly counterbalanced by other forces. We may, however, consider, even in this case, that each force is actually producing its own motion independently of all the other forces, provided that the displacements produced by the several forces are added by the rule shortly to be given.

The *Acceleration* of a particle is the rate of increase of its velocity.

A negative acceleration is often called a *Retardation* or a *Deceleration.*

Rest.—If a particle occupies the same point in space during any finite interval of time, it is said to be at *rest* during that interval.

In reality we can never tell when a particle is at rest. Rest and motion are only relative terms. When we say a particle is at rest, we mean that it has no velocity relative to some body which we consider fixed in space. But in order to be sure that a body is fixed in space we should have to be able to distinguish between similar portions of space. We can, in fact, only perceive the differences in the motions of bodies, and it is with these differences that mechanics is really concerned.

The *Angular Velocity* of a line moving in one plane is the rate of increase, in radians per second usually, of the angle which the line makes with some fixed line in the plane.

The angular velocity of a rigid body about a fixed axis is the rate of increase of the angle which any plane, containing the axis and fixed in the body, makes with a plane also containing the axis and fixed in space.

3. Units.—The units of length and time used in this book will be a foot and a second respectively, unless other units are specially mentioned. The unit of mass will generally be a pound, which is the mass of a piece of platinum preserved by the British Government as a standard.

4. Vectors.—A quantity which cannot be completely determined, without fixing a direction as well as a magnitude, is called a vector quantity. The displacement of a particle, which is the quantity indicating the distance of the particle from some point of reference, is a vector, since both a magnitude (namely, the length of line joining the point of reference to the particle) and a direction are needed for its complete specification. Similarly, velocity and acceleration are vector quantities.

5. A vector quantity can be completely represented by a line. A vector such as a displacement or a force is, moreover, only correctly represented by a particular line. A vector of another kind, such as a couple, can be represented equally well by any line equal and parallel to a given line. Two vectors of the first kind are equal only when they are represented by equal vectors in the same line, whereas two vectors of the second kind are equal when they are represented by equal and parallel lines. Since most of the important properties of vectors are

common to both kinds, we shall assume, for the present, that two vectors are equal if they are represented by equal and parallel lines.

6. Quantities which do not depend on direction, and which are completely specified by magnitude only, are called *scalar quantities*, or simply *scalars*. Time, for example, is a scalar quantity. All non-geometrical quantities are in fact scalars, for it is clear that a vector is a quantity which depends on space.

7. Since displacements are the simplest vectors, we shall prove the important properties of the displacement-vector, and then show that all vectors which are added by the parallelogram rule given in Art. 9 will be subject to the same laws as displacements.

8. Since the displacement from B to A will just annul the displacement from A to B, we have to distinguish between two displacements which are represented by the same line. Many writers use the word "sense" to distinguish between two such displacements. The displacements A to B and B to A are considered by them to be in the same direction but in opposite senses. But in this book we shall, as a rule, use the word "direction" to include sense. When we speak of the direction AB we mean from A to B, and the direction BA shall mean from B to A.

9. If a particle receives, successively, displacements represented by OA and OB, then the final position is the same as if it had received the displacement represented by the diagonal OC of the parallelogram OACB.

If the particle is first displaced from O to A, and then from A to C, its final position is C. Now the displacement AC can be represented by OB. Hence the displacement OC is equivalent to the two displacements OA and OB, since the final position is the same in both cases.

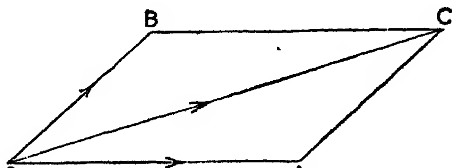

FIG. 1.

10. It is evident that the order in which these displacements are given is immaterial, the final position being the same whether the particle is first displaced from O to A and then from A to C, or first from O to B and then from B to C. The displacement OC is called the *resultant* of the displacements OA and OB.

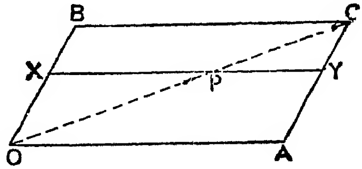

FIG. 2.

11. Even when the particle receives the displacements OA, OB simultaneously, it will arrive ultimately at the position C. We can imagine a particle, P, to be receiving simultaneous displacements in the following way. Suppose the particle is moving along a rod XY which starts from the position OA and moves to the position BC, so that

in every position the rod is parallel to OA. If the particle arrives at Y when the rod arrives at the position BC, the particle will have received the two displacements XY (or OA) and OB simultaneously. Thus, however the displacements are given to the particle, the displacement OC is the resultant of the two displacements OA and OB.

12. The theorem of the preceding paragraph can be put in a different form, which is often more convenient to use. Since the displacement OB is equal to the displacement AC, we see that OC is the resultant of OA and AC. That is, the resultant of the displacements represented by the sides OA and AC of any triangle is represented by the third side OC, the direction of this resultant round the triangle being opposite to the directions of the other two.

13. If a particle receives displacements represented by the sides of an unclosed polygon, not necessarily confined to one plane, the directions of the displacements all leading in the same way round the figure, then the resultant displacement is given by the line joining the beginning of the first to the end of the last side of the polygon.

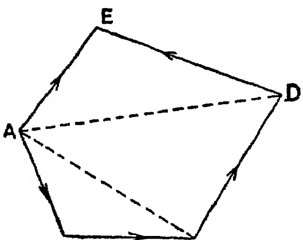

FIG. 3.

This is self-evident from a figure. If a particle receives successively the displacements AB, BC, CD, DE, the effect is just the same as if it had been displaced from A to E. If the displacements are simultaneous, we can use the result proved for two simultaneous displacements to show that the resultant is just the same. For

the resultant of AB and BC is AC;
and the resultant of AC and CD is AD;
i.e. the resultant of AB, BC, and CD is AD:
again, the resultant of AD and DE is AE;
thus the resultant of AB, BC, CD, and DE is AE.

We get a particular case of the preceding theorem when the polygon is a closed one, E and A coinciding. Then the resultant displacement is zero.

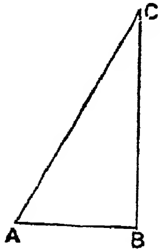

FIG. 4.

14. It will be noticed that the addition of displacements is a different kind of addition from that of ordinary algebra; for in algebra we are concerned only with the magnitudes of quantities, while in the case of displacements the direction of the quantities to be added affects the result. Thus, in the right-angled triangle ABC, where AB = 4 miles, BC = 7 miles, and AC = 8·06 miles, we know that

$$\text{displacement } AB + \text{displacement } BC = \text{displacement } AC.$$

Hence, the sum of these two quantities, whose magnitudes are 4 and 7 miles, is 8·06 miles when added by the vector method. If we ignore direction altogether,

$$AB + BC = 11 \text{ miles.}$$

It is evident that both these methods of summing give results which have rational meanings. If a man walked from A to B and then from B to C, the vector sum gives the man's distance from the starting-point as well as the direction of his position from A; whereas the sum by the second method gives merely the distance walked over by the man, but gives no information about his final position.

15. In future, when we wish to indicate that quantities are to be added by the vector or displacement method we shall put an arrow over the vectors, thus—

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

Using the vector notation, the theorem of Art. 13 concerning the polygon of displacements can be written very simply and concisely—

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} = \overrightarrow{AE}$$

Also

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EA} = 0$$

And as a particular case

$$\overrightarrow{AB} + \overrightarrow{BA} = 0$$

16. Since the sum of $\overrightarrow{AB}$ and $\overrightarrow{BA}$ is zero, each may consistently be regarded as the negative of the other. This gives a meaning to a negative vector. The negative of any vector is the vector obtained by reversing the direction of the first vector and keeping its magnitude the same. In Fig. 5

$$\overrightarrow{AB} - \overrightarrow{BC} = \overrightarrow{AB} + \overrightarrow{CB}$$
$$= \overrightarrow{AB} + \overrightarrow{BD}$$
$$= \overrightarrow{AD}$$

BD being made equal to CB.

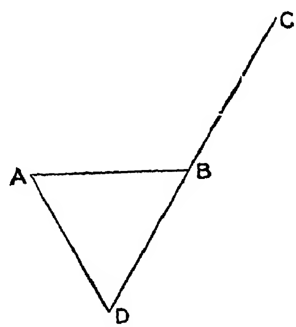

FIG. 5.

17. As a simple application of the subtraction of vectors we will find the displacement of one particle relative to another which has also been displaced. The displacement of one particle relative to another is the displacement which the first would have had to undergo while the second remained fixed in order to leave

the two particles in the same relative position as the actual displacement. If two particles, b and c, starting from A, are displaced to B and C respectively, the displacement of c relative to b is

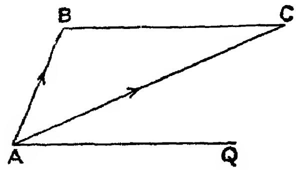

FIG. 6.

$$\overrightarrow{AC} - \overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{BA}$$
$$= \overrightarrow{BA} + \overrightarrow{AC}$$
$$= \overrightarrow{BC}$$

a result which, for this simple case, is obvious without any knowledge of vectors. If AQ is equal and parallel to BC (which is all expressed in the equation $\overrightarrow{AQ} = \overrightarrow{BC}$), then the relative position is just the same as if b had remained at A and c had been displaced to Q.

18. Even when the particles do not start from the same point the relative displacement is given by the same method. The displacements can be represented by lines drawn from the same point, and the relative displacement obtained exactly as before. Thus, if b is moved from A_1 to B, and c from A to C, the displacement of c relative to b is

$$\overrightarrow{AC} - \overrightarrow{A_1B} = \overrightarrow{AC} - \overrightarrow{AD} \text{ (where } \overrightarrow{AD} = \overrightarrow{A_1B})$$
$$= \overrightarrow{AC} + \overrightarrow{DA}$$
$$= \overrightarrow{DC}$$

Let $\overrightarrow{AQ} = \overrightarrow{DC}$. Then the relative positions are obviously the same as if c had been displaced to Q and b had remained at A_1, which shows that $\overrightarrow{AQ}$ or $\overrightarrow{DC}$ represents the relative displacement.

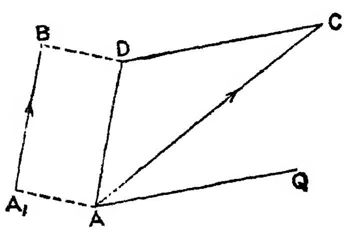

FIG. 7.

19. It follows from the preceding paragraphs that the rule for finding the displacement of a particle c relative to a particle b is the same as if both particles moved in the same straight line, namely, subtract the displacement of b from that of c. This rule includes all cases provided the vector method of subtraction be used.

20. Theorems similar to those we have proved for displacements will be true for any other quantities that are added by the parallelogram method. For all these theorems follow from the parallelogram law.

21. We will now prove that velocities are added by the parallelogram rule.

Firstly, suppose the velocities with which we are dealing are constant. Let a particle have velocities represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$. Then, by the definition of velocity the particle receives, in unit time, displacements represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$. Now, OACB being a parallelogram, $\overrightarrow{OC}$ is the resultant displacement in unit time. But the resultant displacement in unit time is the resultant velocity. Hence OC is the resultant velocity, and it is obtained by the parallelogram rule.

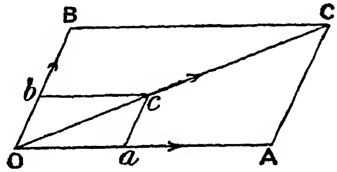

FIG. 8.

22. If the velocities are not constant, consider the displacements in a very short interval of time δt starting from the instant we are considering. In Fig. 8, let O*a*, O*b*, represent OA . δt, OB . δt, which, if the velocities do not vary suddenly by finite amounts, will represent approximately the actual displacements in the interval δt. The resultant of these is O*c* or OC . δt. The resultant velocity is what we obtain on dividing this resultant displacement by δt, namely OC. Thus, whether the velocities be constant or variable, the resultant is obtained by the same process.

23. It is now only necessary to state the theorem in velocities corresponding to the theorem of Art. 13 for displacements.

If a particle has at the same time velocities represented by the sides $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$, $\overrightarrow{DE}$ *of any polygon, then the resultant velocity is represented by* $\overrightarrow{AE}$. *And if E coincides with A the body is at rest.*

24. As an instance of a body having several velocities at the same time, we may mention the case of a man walking on the deck of a ship which is steaming across the current of a river. The resultant velocity of the man is the vector sum of the velocities of the current, of the ship relative to the current, and of the man relative to the ship.

EXAMPLE ON RELATIVE VELOCITY.—*An aviator, who flies with a velocity* v *relative to the air, flies in a straight course from a point A to a point B. If a wind blows with velocity* u *in the direction making an angle* θ *with* $\overrightarrow{AB}$, *find his velocity relative to the earth, and the direction in which he must steer his machine.*

The actual velocity of the machine is the vector sum of the velocity of the air and the velocity v in some direction which will make the sum into a vector along AB.

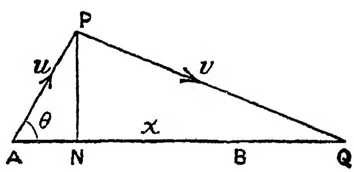

FIG. 8A.

From A draw $\overrightarrow{AP}$ to represent u. Then a vector representing v, the magnitude of which is known, must start at P and end on the line

AB. Therefore describe a circle of radius v and with centre at P, and let Q be the point where it cuts AB on the same side of A as B. Then $\overrightarrow{PQ}$ represents the man's velocity relative to the air, and $\overrightarrow{PQ}$ is the direction in which he must steer. Also $\overrightarrow{AQ}$ is his velocity relative to the earth.

PN is drawn perpendicular to AB.

Then, if x denotes AQ,

$$NQ = \sqrt{(PQ^2 - NP^2)}$$

that is, $$x - u \cos \theta = \sqrt{(v^2 - u^2 \sin^2 \theta)}$$

Therefore $$x = u \cos \theta + \sqrt{(v^2 - u^2 \sin^2 \theta)}$$

which is the required velocity.

To find the direction in which he must steer, we need only find the angle Q.

Now, $$\text{Sin } Q = \frac{u}{v} \sin \theta$$

which gives Q.

25. To show that the parallelogram method of addition applies to accelerations we need only repeat the argument of Art. 21, substituting velocity for displacement and acceleration for velocity, since acceleration bears exactly the same relation to velocity as velocity bears to displacement.

26. We can now state the polygon theorem for accelerations thus—

If a particle has, at the same time, accelerations represented by the sides $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$, $\overrightarrow{DE}$, *of any polygon, then its resultant acceleration is given by* $\overrightarrow{AE}$. *And if E coincides with A the particle has no acceleration; that is, the particle moves with constant velocity.*

27. It will be useful at this stage to call particular attention to the meaning of "constant velocity" in the preceding paragraph, and wherever else it may occur in this book. Two velocities are equal only when they are in parallel directions and equal in magnitude. Consequently, the velocity of a particle remains constant only when the particle continues to move in the same straight line traversing equal distances in any equal intervals of time. A velocity may be changing owing to a change in its direction while its magnitude remains the same. The magnitude of the velocity of a body is called its *speed.* Speed therefore differs from velocity in that the first has magnitude only, while the second has direction as well as magnitude. The velocity of a particle is determined when we are given its speed and the direction of its motion.

28. Suppose a particle has, at any instant, a velocity represented by OA, and, at a time δt later, a velocity represented by OB; then the increase of velocity in this time is—

$$\overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{OB} + \overrightarrow{AO}$$
$$= \overrightarrow{AB}$$

The mean acceleration, therefore, during this interval is—

$$\frac{\overrightarrow{AB}}{\delta t}$$

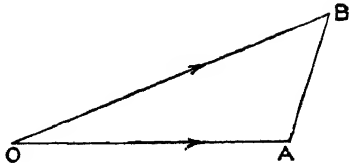

FIG. 9.

and if δt be made infinitely small this becomes the true acceleration at the instant. This quantity is, of course, a vector. It is clear from the foregoing that the direction of the acceleration is entirely independent of the direction of the velocity.

Generally the rate of change of a vector is a vector whose direction in no way depends on that of the first vector.

29. By the aid of one of Newton's laws of motion we can now show that the vector method of addition and subtraction applies to forces. His second law is—

The rate of change of momentum of a body is proportional to the applied force and is in the direction of that force.

Now, the momentum of a body is the product of its mass and its velocity. Hence the rate of change of its momentum is the product of its mass and the rate of change of its velocity. That is, the force producing any acceleration is proportional to the product of the mass and the acceleration.

30. If several forces act on a body at the same time, each force must be considered to be producing its own acceleration independently of the rest. The accelerations produced by the several forces will be equivalent to a single acceleration, namely, their resultant, which could be produced by a single force in the same direction as this resultant. The single force which could produce the same effect as the several forces is the resultant of those forces; and, since the forces are proportional to the accelerations, the lines representing the latter could equally well represent the former. Thus the same polygon which gives the resultant acceleration will give the resultant force; whence it follows that the method of compounding forces is exactly similar to that of compounding accelerations. The vector method of addition and subtraction will therefore apply to forces as well as to accelerations, velocities, and displacements.

31. Not only is it necessary to compound several forces into a single resultant, but it is frequently necessary to perform the reverse operation, namely, to resolve a single force into two or more components. We will show how this is effected.

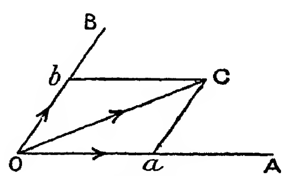

FIG. 10.

32. To resolve the force OC into two components parallel to two given lines OA and OB.

From C draw Ca, Cb, parallel to BO and AO respectively; then $\overrightarrow{Oa}$ and $\overrightarrow{Ob}$ are the components required.

33. The student should be careful to notice that the magnitude of the component along each line depends on the direction of the other. Suppose, for example, that OC and OA remain fixed while OB turns about O so as to increase the angle AOB; as this takes place the point a will move away from O, and when AOB is nearly two right angles Oa, and also Ob, will be very large.

34. The commonest way of resolving forces is along two axes perpendicular to each other. Whenever we speak of the component of a force along a given line in its plane, when no other axis has been mentioned, it must be understood to mean the component obtained by resolving the force along the given line and a line perpendicular to the given line.

35. By the method of obtaining the components of a force along two given axes in its plane it is clear that, for any given pair of axes, there can only be one pair of components of any force. If, however, we wish to resolve a force along three lines in the same plane as the force we shall not get a single set of three components; for we can choose arbitrarily a component of any magnitude along one of these lines, and then resolve the vector difference between the given force and the chosen component along the other two lines.

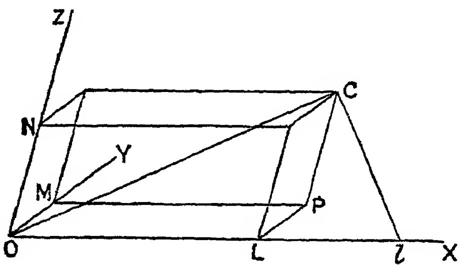

FIG. 11.

36. If a force be resolved along three lines not in the same plane, a single set of three components is obtained.—Let OC be the given force; OX, OY, OZ, the three lines not in the same plane.

Let a parallelopiped be described having its edges parallel to OX, OY, OZ, and having OC as a diagonal. The edges OL, OM, ON, are the components of OC along the given lines. For

$$\overrightarrow{OL} + \overrightarrow{OM} + \overrightarrow{ON} = \overrightarrow{OL} + \overrightarrow{LP} + \overrightarrow{PC}$$

$$= \overrightarrow{OC}$$

37. The three components just obtained are the only possible components along the given lines. For let us suppose that the component along OX were Ol instead of OL. Then the other two components must equal

$$\overrightarrow{OC} - \overrightarrow{Ol} = \overrightarrow{OC} + \overrightarrow{lO}$$

$$= \overrightarrow{lC}$$

Now the resultant of forces along OY and OZ must be in their plane. Therefore $\overrightarrow{lC}$ cannot be equal to this resultant unless it is parallel to the plane YOZ. But unless l coincides with L, $\overrightarrow{lC}$ is not parallel to this plane. Hence l must coincide with L; *i.e.* $\overrightarrow{OL}$ is the only possible component along OX with the given directions of OY and OZ. Similarly, it can be shown that the other two components could not have any other magnitude.

38. If several forces in one plane and their resultant be each resolved into components along two given lines in their plane, the sum of the components of the forces along either of the lines is equal to the component of their resultant along that line.

Let AB, BC, CD, represent the forces; then AD represents their resultant. Let AX, AY, be the two given lines. If BL, CM, DN, be drawn parallel to AY and meeting AX in L, M, and N, then AL, LM, MN, and AN, are the components along AX of AB, BC, CD, and AD. But

$$\overrightarrow{AL} + \overrightarrow{LM} + \overrightarrow{MN} = \overrightarrow{AN}$$

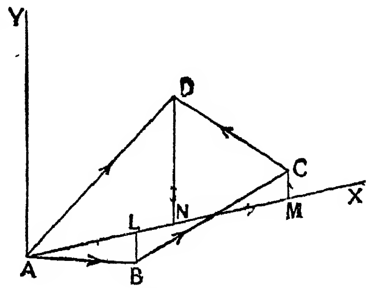

FIG. 12.

i.e. the sum of the components of the forces is equal to the component of their resultant.

39. By means of the preceding theorem we can find analytically the resultant of a system of forces in one plane which act through one point.

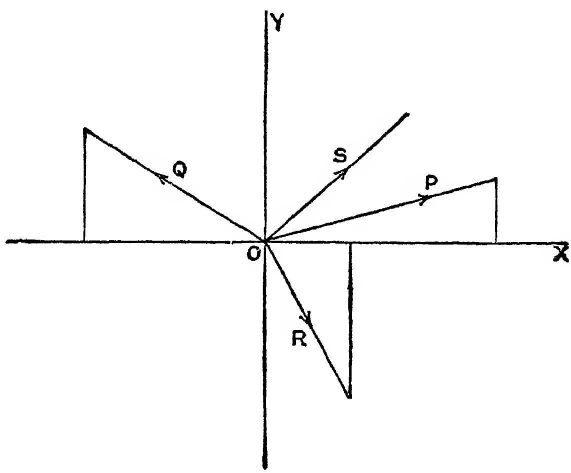

FIG. 13.

Let P, Q, R, be three co-planar forces acting through O whose resultant S is required.

Let OX, OY, be two mutually perpendicular axes in the plane of the forces, and let the angles which the positive directions of P, Q, R, and S, make with OX be α, β, γ, and θ, these angles being considered positive when measured from OX towards OY. Then

component of S along OX = sum of components of P, Q, and R, along OX

i.e.
$$S \cos \theta = P \cos \alpha + Q \cos \beta + R \cos \gamma \quad . \quad . \quad . \quad (1)$$

Similarly, by equating the components along OY,

$$S \sin \theta = P \sin \alpha + Q \sin \beta + R \sin \gamma \quad . \quad . \quad . \quad (2)$$

Squaring both sides of (1) and (2) and adding,

$$S^2 = (P \cos \alpha + Q \cos \beta + R \cos \gamma)^2 + (P \sin \alpha + Q \sin \beta + R \sin \gamma)^2$$

which gives the magnitude of S.

Dividing (2) by (1), we get

$$\tan \theta = \frac{P \sin \alpha + Q \sin \beta + R \sin \gamma}{P \cos \alpha + Q \cos \beta + R \cos \gamma}$$

which gives the direction of S. Thus S is completely determined.

40. A theorem similar to that of Art. 38 can be shown to be true for forces which are not in one plane, when the components are taken along three axes not in the same plane. For it is clear, from Art. 36, that the component of any force OC (Fig. 11) along one of the three axes, OX say, is the intercept cut off OX by two planes through O and C parallel to the plane of the other two axes. Now, in Art. 38, if A, B, C, and D, are not in one plane, and components are taken along OX and two other axes, instead of lines through B, C, and D, parallel to OY as in Art. 38, we shall have planes parallel to the other two axes. The proof that the component of AD, the resultant, is equal to the sum of the components of AB, BC, and CD, along OX is, after this point, just the same as for co-planar forces.

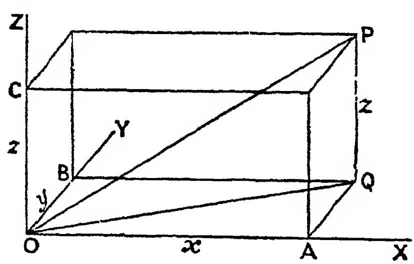

FIG. 14.

41. To find analytically the resultant of a system of forces not all in the same plane, resolve them along three mutually perpendicular axes OX, OY, OZ. Let x, y, and z, be the sum of the components of all the forces along these axes. Then the resultant, R, of the forces is the resultant of x, y, and z.

In the figure OA $= x$, OB $= y$, OC $= z$, and OP $=$ R.

$$\text{Now} \qquad x^2 + y^2 = OQ^2$$
$$\text{Therefore} \qquad x^2 + y^2 + z^2 = OQ^2 + QP^2$$
$$= OP^2$$
$$= R^2$$

Thus the magnitude of R is found.

The direction of R is determined, because we know that it makes angles with the axes OX, OY, OZ, whose cosines are—

$$\frac{x}{R}, \frac{y}{R}, \frac{z}{R}$$

42. So far we have not introduced the derived units of velocity, acceleration, and force, all the properties we have proved being independent of the particular unit chosen. We will now introduce definite units.

Taking a foot and a second as the units of length and time, the simplest unit of velocity is a velocity of one foot per second. If a body moves in a straight line at a uniform rate over v feet in each second, then the magnitude of its velocity is v feet per second. If a body, moving in a straight line, has not a uniform velocity, but has a displacement s feet from some reference point in the line of motion at the end of t seconds from a particular instant, then its velocity is $\frac{ds}{dt}$ feet per second. This general form for the velocity holds, naturally, for the simplest case of uniform motion as well as for variable motion.

43. **Fluxional Notation.**—Differential coefficients with respect to time, that is, rates of increase with respect to time, were indicated by Newton simply by putting a dot over the differentiated quantity, and this notation is still frequently used in dynamics, where it is often very convenient. Thus, $\dot{x}$ and $\frac{dx}{dt}$ are different symbols for the same quantity. Likewise $\ddot{x}$ means $\frac{d^2x}{dt^2}$, but the notation would clearly get awkward for differential coefficients of high orders.

To indicate differential coefficients with respect to quantities other than time, we have to make use of the fact that

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{\dot{y}}{\dot{x}}$$

In dynamics, however, it is nearly always differentiations with respect to time that are used, and consequently there is little or no need of combinations of the fluxions.

44. If v is the velocity of a body in feet per second moving along a straight line, this velocity being regarded as a function of the number of seconds t after some particular instant, the acceleration of the body is $\frac{dv}{dt}$. In the case of uniform acceleration $\frac{dv}{dt}$ is the increase in velocity per second; that is, it is the number of feet per second of

velocity gained by the body per second. Thus it will be seen that the time-unit is involved twice in acceleration; so the unit of acceleration is one foot per second per second.

45. Newton's second law states that if a force F gives an acceleration f to a mass of m pounds, then

$$F \propto mf$$

Now it will clearly simplify matters—in dynamics at least—if we so choose the unit of force that

$$F = mf$$

The unit force in this system is the force which gives to a mass of one pound an acceleration of one foot per second per second. This is the unit used in dynamics, and it is called a *Poundal*. But in statics another unit is used, an accidental unit, which will now be explained.

46. Experience tell us that all bodies, raised above the surface of the earth and then let go, fall towards the earth. Accurate experiments show that all bodies *in vacuo* near the earth's surface, fall towards the earth with an acceleration of 32·2 feet per second per second. And further, Newton's laws, which are the outcome of experience, tell us that a force is needed to produce acceleration in a body. Hence the earth must be pulling each pound of matter with a force of 32·2 poundals. The force with which the earth pulls any body near its surface is called the *weight* of the body. The weight of one pound, which, as we have seen, is 32·2 poundals, is the unit of force commonly used in statics, and it is called a *force of one pound* or *one pound weight*. There is a wide distinction between one pound mass and one pound weight; the first denotes a quantity of matter, and the second, the force with which the earth pulls that matter, an entirely accidental force, since it depends not only on the mass but also on the size of the earth. For this reason the poundal is called the *Absolute Unit* of force, and the pound is the *Gravitation Unit* of force.

The acceleration of a body falling near the earth's surface *in vacuo* is denoted by g.

47. Masses are compared in practice by comparing their weights If two masses have equal weights they are considered to be equal masses. But what we do really know about the masses is that the earth exerts equal forces on them; and this, combined with the knowledge that both these bodies will fall with the same acceleration under the action of these forces, tells us that equal forces produce equal accelerations in them. Now, by Newton's second law, since the forces and the accelerations are the same in the two cases, the masses must be the same. But Newton's law is not a proof that the masses are equal, but an assumption. In the case of bodies composed of exactly the same substance this assumption can be verified in other ways; but for bodies composed of different substances, such as iron and copper, the assumption can only be taken as a definition of equal masses.

48. In the metric system the units of length, mass, and time, are a centimetre, a gram, and a second, respectively The force which gives

to one gram an acceleration of one centimetre per second per second is called a dyne. Since one foot = 30·48 centimetres, the acceleration of falling bodies at the earth's surface is 32·18 × 30·48 or 981 centimetres per second per second. The relations between the derived units in the two systems will be investigated in the chapter on *Units and Dimensions*. The *weight of one gram* is the unit of force commonly used in statics when metric units are employed.

EXAMPLES ON CHAPTER I

1. A particle moves round a circle of 3 feet radius at a uniform speed of 4 feet per second. What is its displacement from the starting-point at the end of π seconds? What is its average velocity in this interval, and what angle does this velocity make with the initial velocity?

$$\left[3\sqrt{3} \text{ feet}; \ \frac{3\sqrt{3}}{\pi} \text{ feet per sec.}; \ 120^\circ.\right]$$

2. P and Q are any two points on the rim of a rotating flywheel. Find the velocity of Q relative to P, given the angular velocity ω of the wheel, and show that it is the same as if the wheel were rotating about P with the same angular velocity.

3. P and Q are any two points in a plate rotating about an axis perpendicular to its plane with a given angular velocity. Show that the relative velocity is the same for all positions of the axis of rotation.

4. Show that the velocity of a point P on a spoke of a wheel rolling along a road with an angular velocity ω, is CP. ω at right angles to CP, where C denotes the point of the wheel in contact with the road.

[Add the vectors representing the velocity of the centre and the velocity of P relative to the centre.]

5. Show that the sum of the vectors represented by the medians of any triangle, all drawn from the angular points towards the opposite sides, is zero.

6. Forces represented by $m.\overrightarrow{\text{OA}}$ and $n.\overrightarrow{\text{OB}}$ act along OA and OB. Show that the resultant is $(m+n)\overrightarrow{\text{OG}}$, where G is the point in AB such that $m.\text{AG} = n.\text{GB}$.

$$[\textit{Proof}—\quad m.\overrightarrow{\text{OA}} = m(\overrightarrow{\text{OG}} + \overrightarrow{\text{GA}}) = m.\overrightarrow{\text{OG}} - m.\overrightarrow{\text{AG}}$$

$$n.\overrightarrow{\text{OB}} = n(\overrightarrow{\text{OG}} + \overrightarrow{\text{GB}}) = n.\overrightarrow{\text{OG}} + m.\overrightarrow{\text{AG}}$$

$$\text{therefore} \qquad m.\overrightarrow{\text{OA}} + n.\overrightarrow{\text{OB}} = (m+n)\overrightarrow{\text{OG}}.]$$

7. ABCD is a parallelogram, and G the intersection of the diagonals. O is any point inside or outside the plane of the parallelogram. Show that the resultant of forces represented by $\overrightarrow{\text{OA}}$, $\overrightarrow{\text{OB}}$, $\overrightarrow{\text{OC}}$, $\overrightarrow{\text{OD}}$, is $4.\overrightarrow{\text{OG}}$.

8. ABC is any triangle, G the intersection of the medians, O any point inside or outside the plane of the triangle. Prove that the sum of the vectors $\overrightarrow{\text{OA}}$, $\overrightarrow{\text{OB}}$, $\overrightarrow{\text{OC}}$, is $3.\overrightarrow{\text{OG}}$.

9. Vectors are represented in magnitude and direction by lines drawn

from any point O to the eight corners of a cube. Show that their sum is represented by $8 \cdot \overrightarrow{OG}$, where G is the centre of the cube.

10. A, B, C, D, etc., are the corners of a regular polygon (or a regular polyhedron). If the number of corners is n, show that the sum of the vectors drawn from any point O to all the corners is $n \cdot \overrightarrow{OG}$, where G is the centre of the figure.

11. Show that the sum of the vectors represented by the sides $\overrightarrow{AB}$, $\overrightarrow{DC}$ of any quadrilateral ABCD is equivalent to the sum of the vectors represented by the diagonals $\overrightarrow{AC}$, $\overrightarrow{DB}$.

12. Across a river a quarter of a mile wide, which flows at he rate of one mile per hour, a man rows in a straight course from one bank to a point one-eighth of a mile up-stream on the opposite bank. If his rate of rowing relative to the water is 2 miles an hour, how long does he take?

[$12\frac{1}{2}$ minutes.]

13. Suppose an airplane travels in still air at 200 miles an hour. How long will it take the aviator to fly round a square course whose side is twelve miles long if there is a wind blowing parallel to a pair of sides at the rate of 56 miles an hour?

[$15\frac{5}{18}$ minutes.]

14. How long would the aviator mentioned in the last question take to fly round the square course if the wind were blowing parallel to a diagonal of the square?

[15·31 minutes.]

15. An aviator, who flies in still air at 114 miles per hour, flies round a course in the form of an equilateral triangle with sides twelve miles long, while a wind blows parallel to one side at 30 miles per hour. Show that he takes 20 minutes to go round the course.

16. If an aviator flies round a triangular course, each side of which is c miles long, while the wind blows at u miles per hour parallel to a side, show that he takes

$$\frac{c\{v + \sqrt{(4v^2 - 3u^2)}\}}{v^2 - u^2} \text{ hours}$$

to complete the circuit in either direction, v being his velocity relative to the air.

17. In what direction must a boat be steered across a river which flows at 3 miles per hour, by a man rowing at 4 miles per hour, in order to make a course at right angles to the banks?

[At $\sin^{-1}(\frac{3}{4})$ up-stream.]

18. An observer on a steamer which is travelling due west at 14 miles an hour, notices another steamer travelling south-east. By observations on the distances of the second steamer at different times he calculates that its velocity relative to the first steamer is 26 miles an hour. What is the true velocity of the second steamer?

[$10\sqrt{2}$ miles per hour.]

19. ABC is a triangle with AB horizontal and C above AB. F is the foot of the perpendicular from C on AB. From F perpendiculars are drawn to AC and BC, meeting them in P and Q. Now suppose bodies slide down CA and CB with accelerations $g \sin A$, $g \sin B$. Show graphically that the relative acceleration is parallel to PQ.

CHAPTER II

FIRST PRINCIPLES OF STATICS

49. At this point we shall need a few more terms, which we will now define.

A *Rigid Body* is a body which maintains its shape and size under the action of any forces. Or better, a rigid body is a body in which the distance between every pair of particles remains constant under the action of any forces.

Like many other things in mechanics, an absolutely rigid body does not exist. All bodies alter their shape or size or both under the action of forces. If the two ends of a rod be pulled away from each other the rod is lengthened, and if the ends be pushed towards each other the rod is shortened. But this alteration in length is so small for ordinary forces that it may be neglected in most cases. For this reason we shall treat all bodies as if they were rigid, except in the questions on elasticity, where the alteration in shape is the subject specially considered.

Tension and Thrust.—If forces act on the ends of a rod or string tending to pull them apart, the rod or string is said to be in *tension.* If the forces on a rod tend to push the ends together, the rod is said to be in *thrust.*

Action and Reaction.—When a body A, exerts a force on a body B, observation tells us that B exerts a force on A in the opposite direction. Newton's third law of force states that these two forces are equal and opposite; that is, their magnitudes are equal, and they act in opposite directions in the same straight line. Either of these forces is called the *action* of one body on the other, and the other force is called the *reaction.*

The conviction that action and reaction are equal is not arrived at directly by observation. This conviction is forced upon us by the universal agreement between the deductions drawn from the principle and our experience.

Smooth Bodies.—A body is said to be smooth when the only force that can be exerted at any point of its surface, by another body touching it there, is a force along the common normal to the surfaces of the two bodies at that point.

Bodies that are not smooth are called *rough* bodies. No body is perfectly smooth. A smooth body is just as unreal as a rigid body, but both have their uses in mechanics.

Equilibrium.—A system of forces are in *equilibrium* when the effect which they would produce on a rigid body is nil.

Sometimes we speak of the body on which the forces act as being in equilibrium. This means simply that the forces are in equilibrium.

50. By the methods of the last chapter the vector representing the sum of a number of forces can be very easily obtained. We have only to draw a series of vectors representing the forces, each vector beginning at the end of the last one, and then the vector completing the polygon represents the resultant or sum. But a force has a definite line of action as well as magnitude and direction, and the above process does not give the line of action of the resultant. In the algebra of vectors we make no distinction between vectors which are equal and parallel, but in dealing with forces we are obliged to make a distinction. We may take it as an axiom that two forces acting on a rigid body cannot balance each other unless they act in opposite directions along the same straight line and have equal magnitudes. Hence it follows that two forces are only truly equal when they act in the same direction along the same straight line and are equal in magnitude.

51. We could, however, apply the parallelogram method of addition to find the line of action as well as the vector sum of several forces. For, since it is evident that the resultant of two forces passes through their point of intersection, it follows that the diagonal of the force-parallelogram gives the resultant in its actual position. To find the resultant of several forces it is only necessary, therefore, to sum two of the forces, then add the resultant of these to one of the remaining forces, and so on until the whole are reduced to a single force.

52. We must mention here an assumption which we shall constantly make in the course of the work. It is known by the name of "*The Transmissibility of Force.*" The assumption is that a force applied at any point of a rigid body can be balanced by a force of equal magnitude acting in the opposite direction, and applied at any point in the line of action of the first force. Thus a pull applied at one end of a rod can be balanced by a pull applied at the other end, and when the rod is in equilibrium under such a pair of forces, the lines of action of the forces both run along the rod. Likewise two equal thrusts applied at the ends of a rod can balance each other. An immediate consequence of the above assumption is that, in dealing with forces in equilibrium, we may suppose any force to act at any point in its line of action without disturbing the equilibrium. It will appear, when we deal with the subject of elasticity, that the stresses due to equal forces applied at different points are by no means the same. Moreover, the conditions of stability are quite different. The thing which is really the same for the different forces is their capacity, as long as the equilibrium is maintained, to balance other forces acting on the rigid body.

53. If all the forces acting on a body pass through one point, the resultant will clearly pass through the same point; consequently, when the magnitude and direction of the resultant is found, this resultant is completely determined.

54. In trying to find the line of action of the resultant of several

forces by the parallelogram method, it might happen that we arrive at a point where the original set of forces are reduced to several parallel forces. At this point we could carry the parallelogram method no further. We will now give a method of finding the resultant of two parallel forces. By several successive applications of this method we could, of course, find the resultant of any number of parallel forces.

55. There are two cases to consider, namely, (i) where the two forces act in the same direction, and (ii) where the two forces act in opposite directions. But one method will apply to both cases. We give a figure for each case.

Let $\overrightarrow{AB}$, $\overrightarrow{CD}$ represent the two forces whose resultant is required. Join AC, and at A and C introduce two equal and opposite forces, $\overrightarrow{AP}$, $\overrightarrow{CQ}$, in the line AC. Since these two forces balance each other the resultant of all the four forces will be the resultant of the two given forces. By the parallelogram method we find the resultant $\overrightarrow{AR}$ of $\overrightarrow{AB}$ and $\overrightarrow{AP}$; also the resultant $\overrightarrow{CS}$ of $\overrightarrow{CD}$ and $\overrightarrow{CQ}$. These two resultants will meet in some point M if $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are not equal and in opposite directions.

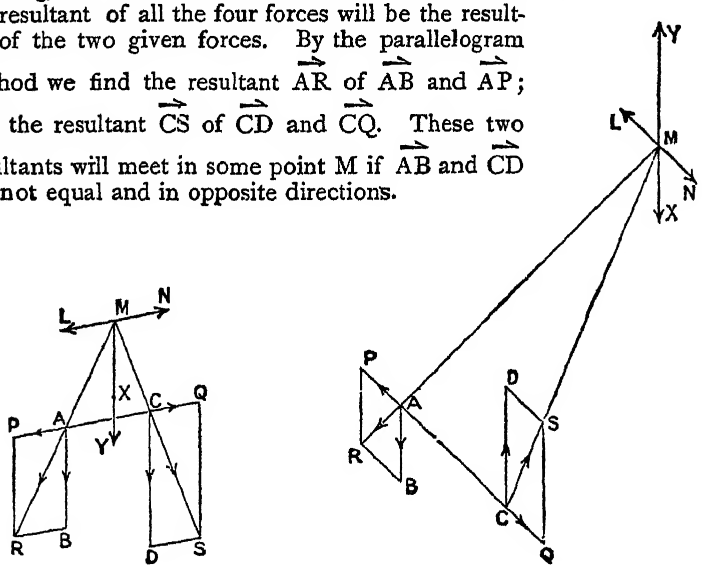

FIG. 15.

Now, the force $\overrightarrow{AR}$ may be considered to act at M, and it is equivalent to $\overrightarrow{ML}$ and $\overrightarrow{MX}$ where $\overrightarrow{ML} = \overrightarrow{AP}$ and $\overrightarrow{MX} = \overrightarrow{AB}$. Also the force $\overrightarrow{CS}$ is equivalent to $\overrightarrow{MN}$ and $\overrightarrow{MY}$ where $\overrightarrow{MN} = \overrightarrow{CQ}$ and $\overrightarrow{MY} = \overrightarrow{CD}$. The resultant of these four forces acting at M is clearly the vector sum of $\overrightarrow{MX}$ and $\overrightarrow{MY}$, that is, the vector sum of $\overrightarrow{AB}$ and $\overrightarrow{CD}$, and it acts through M. The resultant of two parallel forces is just their vector sum exactly as if the forces were not parallel, the only difficulty being in finding the line of action of the resultant.

If the forces are in opposite directions and equal in magnitude the diagonal RA and CS will clearly be parallel, and the construction will fail. There is, in fact, no single resultant of two equal but opposite parallel forces. They can only be left as a pair of parallel forces. Such a pair of forces form what is called a *couple.*

By adding forces two at a time we can always reduce any system of parallel forces to a single force or a single couple. It follows from this and from previous theorems that any system of forces whatever can be reduced to a single force or a single couple.

56. Moment of a force.—The moment of a force about any point in its plane is the product of the force and the perpendicular distance of the point from the line of action of the force. Thus the moment of the force F about the point P is pF.

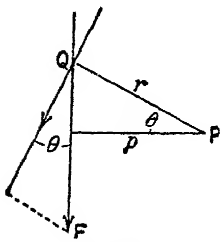

FIG. 16.

We can derive another very convenient expression for the moment of a force as follows :—

Let Q be any point on the line of action of F, and let PQ = r. Then the moment of F about P

$$= pF$$
$$= r\cos\theta \cdot F$$
$$= r \times \text{the component of F perpendicular to PQ,}$$

for F cos θ is the component of F perpendicular to PQ if the other component is along PQ.

57. We have defined the moment of a force about a point as being the product of the force and the perpendicular distance of the point from the force. Now we know that the force F reversed in direction would exactly balance F, and whatever effect one of the forces would produce would be exactly annulled by the effect of the other. It is reasonable, then, that the moment of one of these forces should be considered to be the negative of the moment of the other; and this convention is a reasonable one no matter what effect is measured by the moment of a force. It is proved, in fact, in Chapter XVIII., that the moment of a force about any fixed axis measures the turning effect of the force about that axis. The usual convention is to consider a moment positive when the effect of the force would be to turn the body in the direction opposite to that of the hands of a clock (the face being upwards), and negative in the other case.

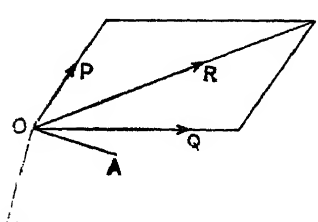

FIG. 17.

58. The sum of the moments of any number of co-planar forces about any point in their plane is equal to the moment of their resultant about that point.

We will first prove the theorem for two forces which meet at a point.

Let P and Q be the two forces

meeting at O, R their resultant, and A the point about which the moments are to be taken. Also let OB be drawn perpendicular to OA, the direction of OB being the same as that of a force which would have a positive moment about A. Then

$$\left.\begin{array}{l}\text{Sum of moments of P and Q}\\ \quad\text{about A}\end{array}\right\} = \text{OA} \times \begin{array}{r}\text{sum of components of P and}\\ \text{Q along OB}\end{array}$$
$$= \text{OA} \times \text{component of R along OB}$$
$$= \text{moment of R about A.}$$

NOTE.—If OA or AO produced lies inside the parallelogram, the components will have opposite signs. But the proof is exactly the same as above. The negative component has a negative moment, and it must be considered negative in summing.

59. We will now prove the theorem for a pair of parallel forces which do not form a couple.

Let P and Q be the parallel forces, R their resultant, and A the point about which the moments are to be taken. As in finding the resultant of two parallel forces, let us introduce a pair of equal but opposite forces T and −T along the same line EC. Let V be the resultant of P and T, and W the resultant of Q and −T.

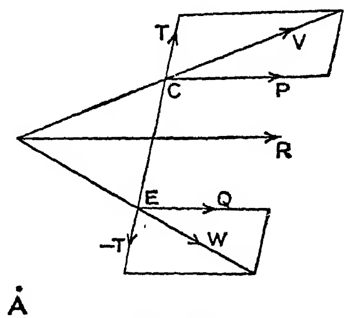

FIG. 18.

Now, by Art. 58

$$\left.\begin{array}{l}\text{sum of moments of P and T}\\ \quad\text{about A}\end{array}\right\} = \text{moment of V about A.}$$

Also

$$\left.\begin{array}{l}\text{sum of moments of Q and }-\text{T}\\ \quad\text{about A}\end{array}\right\} = \text{moment of W about A.}$$

Again,

$$\left.\begin{array}{l}\text{sum of moments of V and W}\\ \quad\text{about A}\end{array}\right\} = \text{moment of R about A.}$$

Hence

$$\left.\begin{array}{l}\text{sum of moments of P, T, Q,}\\ \quad -\text{T about A}\end{array}\right\} = \text{moment of R about A.}$$

But clearly the sum of the moments of T and −T is zero. Therefore we arive at the conclusion that the sum of the moments of P and Q about any point is equal to the moment of their resultant about the same point.

60. The resultant of several co-planar forces can be obtained by

several successive operations, each of which consists in adding together two forces. And since the moment of the resultant of each operation about any point is equal to the sum of the moments of the forces giving this resultant, it follows that the moment of the final resultant of all the original forces is equal to the sum of the moments of all those forces. Thus the general theorem is proved.

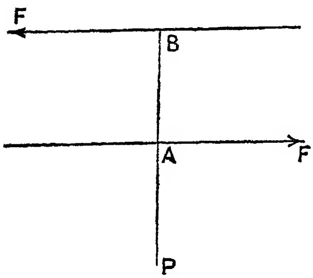

FIG. 19.

61. The moment of a couple about every point in its plane is the same.

Let F be the magnitude of the forces forming the couple, P any point in its plane, PAB a line perpendicular to the forces. Then

$$\begin{aligned}\text{moment of couple about } \mathbf{P} &= \mathrm{F}\,.\,\mathrm{PB} - \mathrm{F}\,.\,\mathrm{PA}\\ &= \mathrm{F}\,.\,\mathrm{AB}\\ &= p\mathrm{F}\end{aligned}$$

where p is the perpendicular distance between the lines of the forces, and is called the arm of the couple. The product pF is independent of the position of **P**; hence the moment is the same about all points in the plane of the couple.

62. Two couples in the same plane the sum of whose moments is zero are in equilibrium.

Firstly, let the forces of the two couples intersect. Let the forces of one couple act along AB and CD, and those of the other couple along AD and CB.

By choosing a convenient scale to represent our forces we can take AB and CD to represent the forces along those lines. Then the moment of the couple formed by these forces is represented by the area of the parallelogram ABCD. But the magnitude of the moment of the second couple is equal to that of the first. Hence the area of the parallelogram ABCD represents the moment of the second couple. It follows from this that the lines AD and CB must represent the magnitude of the forces. Thus our two couples are represented by the four forces $\overrightarrow{\mathrm{AB}}$, $\overrightarrow{\mathrm{AD}}$, $\overrightarrow{\mathrm{CB}}$, $\overrightarrow{\mathrm{CD}}$.

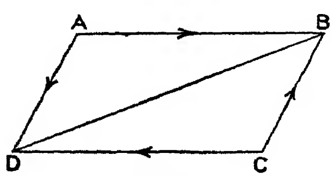

FIG. 20.

Now

$$\overrightarrow{\mathrm{AD}} + \overrightarrow{\mathrm{CD}} = \overrightarrow{\mathrm{BD}}$$

and

$$\overrightarrow{\mathrm{CB}} + \overrightarrow{\mathrm{AB}} = \overrightarrow{\mathrm{DB}}$$

Thus the resultant of the two forces meeting at B is exactly the negative of the resultant of the two forces meeting at D, and these two resultants have the same line of action. Hence they are in equilibrium; that is, the two couples are in equilibrium.

We will now deal with the case of two couples formed by four parallel forces.

We will denote forces in one direction by positive signs and forces in the other direction by negative signs.

Let P and $-P$ be the forces of one couple, Q and $-Q$ those of the other.

The resultant of P and Q is a parallel force $(P+Q)$ through some point C. The resultant of $-P$ and $-Q$ is another parallel force $-(P+Q)$, which also passes through C, as we will now prove.

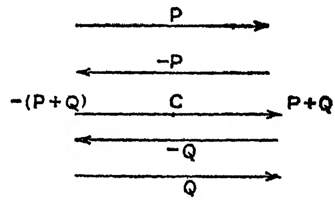

FIG. 21.

The sum of the moments of P and Q about C is equal to the moment of their resultant, which moment is zero. Also the sum of the moments of the four forces P, Q, $-P$, $-Q$ is zero by hypothesis. Hence the sum of the moments of $-P$ and $-Q$ about C must be zero, and since the resultant itself is not zero, it must be the perpendicular distance that is zero. That is, the resultant of $-P$ and $-Q$ passes through C, and therefore it is in equilibrium with the force $(P+Q)$ through the same point. Thus the two couples are in equilibrium.

63. We have now proved that two couples in the same plane, the sum of whose moments is zero, are in equilibrium. Either of these couples may therefore be regarded as the negative of the other, and either of them reversed may be regarded as the exact equivalent of the other. Thus two couples acting in the same plane on a rigid body are exactly equal if they have equal moments.

64. Any two couples acting in parallel planes on a rigid body are equal if they have equal moments.

Up to the present we have usually spoken about moments of forces about *points* in their planes. Since couples are connected with rotations, and rotations take place about *lines* and not about points, we must take moments about lines if we wish to attach a real physical meaning to the moment of a couple. When we say that the moments of two couples in parallel planes are equal we mean that the moments of the couples about any line perpendicular to their planes are equal. That is, the magnitudes of the moments are the same, and they would cause rotation in the same direction. This subject is treated thoroughly in Chapter XXIII.

Let F, $-F$ be the forces of one couple, and let equal and parallel forces be taken in the other plane, forming a similar couple, but with a moment of opposite sign. Let AB and CD be the arms of these couples. These arms will be equal since the moments are equal in magnitude, and they will be parallel. We will show that these two couples are in equilibrium.

The resultant of F at A and F at C is 2F at O, the mid-point of AC. Similarly the resultant of $-$F at B and $-$F at D is $-$2F at the mid-point of DB. But since AB and CD are equal and parallel, AC and BD are the diagonals of a parallelogram, and therefore their mid-points coincide. Thus the two forces 2F and $-$2F act at the same point, and therefore they are in equilibrium. Hence the two couples are in equilibrium.

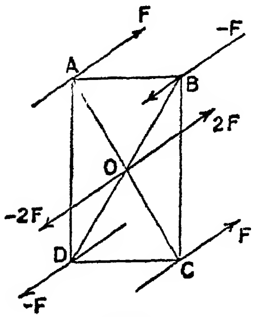

FIG. 22.

Thus the couple whose forces pass through C and D balances the first given couple. But this same couple through C and D will balance the second given couple which is in the same plane and has an equal moment but of opposite sign. Thus, since the two given couples will balance the same couple, they are equal.

65. Since the effects produced by two couples in the same plane or parallel planes are equal if their moments are equal, it follows that the effect of a couple is measured by its moment. The effect produced by several couples in the same plane or in parallel planes is therefore measured by the sum of their moments, the proper sign being, of course, attached to each moment.

66. It has been shown that two couples, acting in the same plane or in parallel planes, are equivalent if they have equal moments. Equal couples, then, have two things in common: namely, equal moments and a common direction of the normals to their planes. These suggest that we might represent a couple by a length measured along a normal to its plane; that is, it seems as if the moment of a couple may be subject to vector laws. Let us examine if this is the case.

67. Two couples, whose moments are equal in all but sign, can be represented by equal lengths along the same line, but these lengths must, for distinction, be drawn in opposite directions. We shall there-

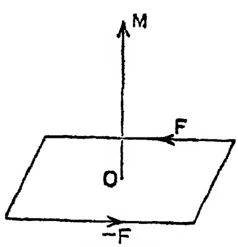

FIG. 23.

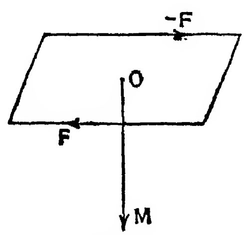

FIG. 24.

fore need a rule concerning the direction in which a vector must be drawn in order to represent the moment of a couple. The accepted rule is as follows: If the couple were acting on a right-handed screw working in a fixed nut, the axis of the screw being perpendicular to

the plane of the couple, the vector must be drawn in the direction in which the axis of the screw would move in consequence of the rotation given by the couple.

In the two figures 23 and 24, F and $-F$ represent the forces of a couple, and OM the moment of the couple in each case.

68. Couples in inclined planes.

Now suppose two couples with moments M_1 and M_2 act in two intersecting planes ABCD and CDEF. We know we can replace any couple by any other couple in the same plane having the same moment. Let us therefore replace the couple in the plane ABCD by a couple whose forces are $-P$ along CD and P along a parallel line AB, the position of which is chosen so as to make the moment the same as the moment of the given couple. Similarly, let the other couple be replaced by P along CD and $-P$ along the parallel line FE. This arrangement is always possible. We have only to choose the lengths AD and DE properly. The sign of the couple in the plane CDEF is regulated by putting E on one or the other side of D.

Now the forces along CD are in equilibrium, and we are thus left with the two forces P and $-P$ along BA and FE respectively, forming

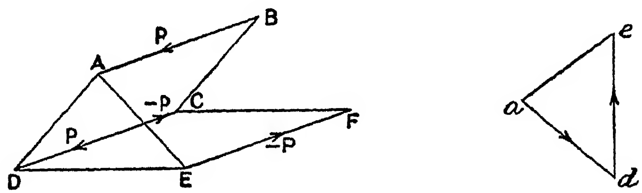

FIG. 25.

a couple in the plane ABFE. Hence the resultant of the two given couples is another couple in a plane inclined to the planes of both. We will now show that this result could have been obtained by adding the vectors which, according to our rule, represent the given couples.

69. In fixing the positions of A and E we were quite at liberty to make AD and DE each perpendicular to CD. Let us suppose this has been done. Then the lines AD, DE, and AE are the arms of the two given couples and of the resultant couple. Since the forces of the couples are all equal it follows that these arms are proportional to the moments of the couples to which they belong.

Now let $\overrightarrow{ad}$, $\overrightarrow{de}$ be drawn to represent the moments of the given couples. These lines will be perpendicular to the planes of the couples, and therefore perpendicular to AD and DE respectively. Also since AD and DE are proportional to the moments of the couples we get

$$\frac{ad}{de}=\frac{\mathrm{AD}}{\mathrm{DE}}$$

Also the angle *ade* is equal to the angle ADE. Hence the triangle *ade*

is similar to the triangle ADE. Therefore $ae : ad : de = \mathrm{AE} : \mathrm{AD} : \mathrm{DE}$. Also $\overrightarrow{ae}$ is perpendicular to the plane AEFB. It follows that ae represents the couple whose arm is AE, that is, the couple which we have proved to be the resultant of the given couples. But this is exactly what we should have got by adding the two vectors $\overrightarrow{ad}$ and $\overrightarrow{de}$. Hence the moments of couples are added by vector rules, provided they are represented in the manner shown above.

70. It follows from what we have just proved that several couples acting on the same rigid body are equivalent to a single couple whose moment is the vector sum of the moments of the several couples. Couples in parallel planes are included in this.

71. Any force acting on a rigid body can be replaced by a parallel force through any chosen point, together with a couple in the same plane as the given force and the point.

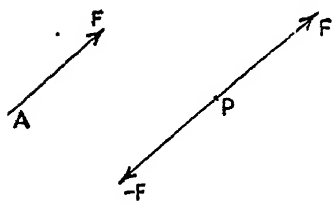

FIG. 26.

Let F be the given force acting through A, and let P be any chosen point.

Let two forces F and $-$F parallel to the given force be introduced at P. These will produce no effect on the rigid body since they balance each other. Hence the two forces at P and the one at A are equivalent to the given force at A. But the forces F at A and $-$F at P form a couple. Thus the given force at A is equivalent to the couple so formed, together with the force F at P.

The moment of the couple is the moment of F at A about P.

72. To find analytically the resultant of a given system of coplanar forces.

Let perpendicular axes OX, OY be taken in the plane of the forces. And suppose that forces whose components parallel to the axes are X_1 and Y_1, X_2 and Y_2, X_3 and Y_3, etc., act at points (x_1, y_1), (x_2, y_2), (x_3, y_3), etc.

The force X_1 acting at $(x_1 y_1)$ can be replaced by X_1 along OX, together with a couple whose moment is $-y_1X_1$. Similarly, the force Y_1 acting at (x_1y_1) can be replaced by Y_1 along OY and a couple whose moment is x_1Y_1. All the other forces can be treated in the same way. Then the given forces are equivalent to

(1) A set of forces X_1, X_2, X_3, etc., along OX;
(2) A set of forces Y_1, Y_2, Y_3, etc., along OY;
(3) A number of couples whose moments are

$$(x_1Y_1 - y_1X_1),\ (x_2Y_2 - y_2X_2), \text{ etc.}$$

Let ΣX denote $X_1 + X_2 + X_3 +$ etc.,
and ΣY denote $Y_1 + Y_2 + Y_3 +$ etc.,
and $\Sigma(xY - yX)$ denote $(x_1Y_1 - y_1X_1) + (x_2Y_2 - y_2X_2) +$ etc.

Now, the forces through O reduce to a single force R through O, making an angle θ with OX such that

$$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2}$$

$$\tan\theta = \frac{\Sigma Y}{\Sigma X}$$

And the couples are equivalent to a single couple whose moment is

$$\Sigma(xY - yX)$$

We have thus reduced the given forces to a single force and a single couple. These again can be reduced to a single force equal and parallel to R, but acting in a different line. This last force is the resultant of the system.

To find where the resultant force acts, let p denote the distance of its line of action from O. Then

$$pR = \Sigma(xY - yX)$$

whence

$$p = \frac{\Sigma(xY - yX)}{\sqrt{\{(\Sigma X)^2 + (\Sigma Y)^2\}}}$$

It might seem that there are two possible positions for the line of action of the resultant both touching a circle of radius p with its centre at the origin. But the correct position of this line is determined by the condition that the moment of R about O must have the same sign as the moment of the couple $\Sigma(xY - yX)$.

Thus the magnitude, direction, and line of action of R, are completely determined.

73. In particular cases the force R, the couple $\Sigma(xY - yX)$, or both, may be zero. R will be zero if $\Sigma X = 0$ and $\Sigma Y = 0$ simultaneously. If R is zero and the couple is not zero, then the resultant is simply the couple and it cannot be further reduced. But if the couple is zero and R is not zero, the resultant passes through O, for p is zero in this case. If R and the couple are both zero the forces are in equilibrium.

74. The method we have just used will apply just as well to a system of parallel forces as to intersecting forces.

Let a system of parallel forces F_1, F_2, F_3, etc., act on a rigid body. Let the axis of y be taken parallel to the forces. Suppose the forces act at distances x_1, x_2, x_3, etc., from the origin. Then, using the formulæ obtained,

$$R = F_1 + F_2 + F_3 + \text{etc.}$$

$$\tan\theta = \infty$$

whence

$$\theta = 90^\circ$$

Also

$$p = \frac{x_1F_1 + x_2F_2 + x_3F_3 + \text{etc.}}{F_1 + F_2 + F_3 + \text{etc.}}$$

75. Centre of parallel forces.

Suppose a number of parallel forces of given magnitudes act at fixed points in a rigid body. Then, whatever the direction of the forces may be relative to the body, the resultant will always pass through another fixed point which is called the "centre" of the parallel forces.

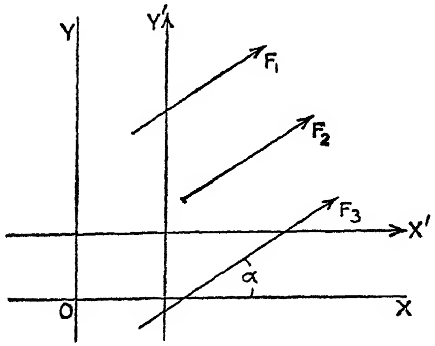

FIG. 27.

Let the forces be F_1, F_2, F_3, etc., acting at points (x_1, y_1), (x_2, y_2), (x_3, y_3) relative to axes fixed in the body, and suppose the direction of the forces makes an angle α with OX. Then we shall prove that the resultant passes through a point which does not depend on α.

Let X′, Y′ denote the sums of the components parallel to OX and OY respectively. Then

$$X' = (F_1 + F_2 + F_3 + \ldots) \cos \alpha$$
$$Y' = (F_1 + F_2 + F_3 + \ldots) \sin \alpha$$

Suppose the line of action of X′ is at a distance $\bar{y}$ from OY, and the line of action of Y′ at a distance $\bar{x}$ from OX. Then, by taking moments about O for the X-components,

$$\bar{y}X' = y_1F_1 \cos \alpha + y_2F_2 \cos \alpha + y_3F_3 \cos \alpha + \ldots;$$

that is, $$\bar{y} = \frac{y_1F_1 + y_2F_2 + y_3F_3 + \ldots}{F_1 + F_2 + F_3 + \ldots}$$

In the same way we get

$$\bar{x} = \frac{x_1F_1 + x_2F_2 + x_3F_3 + \ldots}{F_1 + F_2 + F_3 + \ldots}$$

Now, the resultant of all the given forces is the resultant of X′ and Y′, and since these meet at the point $(\bar{x}, \bar{y})$ the resultant passes through this point. But the values found for $\bar{x}$ and $\bar{y}$ do not involve α, and therefore the point $(\bar{x}, \bar{y})$ is the same for all values of α. Thus the theorem is proved.

76. If some of the forces act in the opposite direction to that of the others, we can allow for this by merely taking these forces to be negative in the formulæ. It need scarcely be pointed out that the magnitude of the resultant is the algebraic sum of the forces.

77. Conditions of equilibrium of a rigid body acted on by forces in one plane.

By experience we know that if a body at rest is acted on by a force or by a couple, the body cannot remain at rest. We can, it is true,

deduce these facts from Newton's laws of motion, and this will be done in the chapters on rigid dynamics. But these laws are themselves the embodiment of experience, and are not so axiomatic as the facts we have just stated. We may as well, therefore, use the above rules as statical axioms.

We have previously found (Art. 72) that a system of coplanar forces can be reduced to a single force or to a single couple. The force will be zero if, with the notation of Art. 72,

$$\Sigma X = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$\Sigma Y = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If the equations (1) and (2) hold the resultant cannot be a force, but it may be a couple. To ensure that the couple is also zero we need only make the moments of all the forces about any point—the origin of co-ordinates for convenience—equal to zero, for this moment is equal to the moment of the couple if there is one. The condition necessary for a zero couple is therefore

$$\Sigma(xY - yX) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Equations (1), (2), and (3) are the necessary and sufficient conditions that the body should be in equilibrium.

78. The three conditions of equilibrium could be put in many different forms, but all these forms could be deduced from the three equations of the last article. If, for instance, the sum of the moments of the forces about any three points not in the same straight line is zero the body will be in equilibrium. For we know the resultant cannot be a couple, because a couple cannot have zero moment about any point in its plane. Also it cannot be a single force, because a force can only have zero moment about points in its line of action, and this line cannot pass through three non-collinear points.

79. Whatever forms, however, the three equations of equilibrium take, we cannot get rid of an equation containing moments. That is, at least one of the equations must be that the sum of the moments about some point is zero, for that is the only way in which we can be sure that the resultant is not a couple. It will be shown in the next article that we may replace the moment equation by an equivalent geometrical condition.

80. Body under three forces.

If a rigid body is in equilibrium under three forces their lines of action must lie in one plane, and they must meet at a point or be parallel.

It is easy to see that this is true; for any one of the forces must be the negative of the resultant of the other two. But this resultant passes through their point of intersection and lies in the same plane. Hence the other force passes through the point of intersection of the two forces and lies in the same plane.

If two of the forces are parallel their resultant will be a force parallel to each of them. The third force has the same line of action as this resultant, and is therefore parallel to them. The case of parallel forces

might have been considered to be a special case of intersecting forces where the point of intersection is at an infinite distance.

81. The condition that the three forces must meet at a point may take the place of the equation of moments in the general conditions of equilibrium in Art. 77. For it is evident that three forces which pass through one point cannot form a couple. In addition to the preceding condition we need only ensure that R should be zero, and this will be the case if each of its components is zero. The condition that the forces are concurrent is, of course, involved in the three equations of equilibrium, and we need not use it in this form unless we choose. In many cases, however, it will be convenient to use the geometrical condition instead of one of the equations of equilibrium.

EXAMPLES ON CHAPTER II

In the following examples the weight of a body which has a symmetrical centre may be taken to be a single vertical force acting at that centre, unless the contrary is stated.

1. A system of coplanar forces reduce to 40 lbs. and 64 lbs. along the axes of x and y respectively, together with a positive couple of magnitude 108 foot-lbs. Where does the resultant meet the axis of x?

[At $x = 1\frac{11}{16}$ feet.]

2. Find the resultant of the following forces and the points where its line of action meets the co-ordinate axes :—

$$\begin{aligned} X_1 &= 4 \text{ lbs., } Y_1 = 7 \text{ lbs., acting at } (2, 1) \\ X_2 &= 10 \text{ lbs., } Y_2 = 5 \text{ lbs., acting at } (4, -1) \\ X_3 &= 0, \ Y_3 = -8 \text{ lbs., acting at } (-5, 6) \\ X_4 &= -12 \text{ lbs., } Y_4 = 1 \text{ lb., acting at } (-7, 2) \end{aligned}$$

[$R = \sqrt{29}$ lbs., and it passes through the points (19·4, 0) and (0, −48·5).]

3. ABC is an isosceles triangle having a right angle at C; forces of 10, 8, and $12\sqrt{2}$ units act from A to C, C to B, and A to B respectively. Find, from the equations of equilibrium, the force that will balance them, and the line along which it will act; also show the four forces in a diagram.

[The resultant has a magnitude 29·73, and acts from P to Q, where P and Q are points on CA, CB respectively such that $CP = \frac{3}{8}CA$, $CQ = \frac{6}{11}CB$.]

4. ABCD is a square of side $2a$, P is the middle point of AD, and Q the middle point of DC; and the following forces act :—

20 from A to B, 20 from C to D, 40 from A to Q, 30 from P to B.

Show that the resultant is a force 50, and that the length of the perpendicular from A on its line of action is about $0{\cdot}27a$. Represent the resultant in a figure.

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5. The ratio of the moments of a certain force about two given points is constant, but the magnitudes of the moments vary. Show that the force always passes through a fixed point on the line joining the given points.

6. The moment of a force about the origin is 60 foot-lbs., and about the point (3 ft., 0) its moment is 24 foot-lbs. Find the point where the force meets the axis of x, and find its component perpendicular to that axis.

[5 feet along x-axis; Y = 12 lbs.]

couples about the co-ordinate axes. The moments of the couples are

$$y_1Z_1 - z_1Y_1 \text{ about OX,}$$
$$z_1X_1 - x_1Z_1 \text{ about OY,}$$
$$x_1Y_1 - y_1X_1 \text{ about OZ.}$$

Any number of forces acting at different points could be treated in the same way. Hence any system of forces whatever can be reduced to three forces ΣX, ΣY, ΣZ along OX, OY, OZ, and three couples about the co-ordinate axes whose moments are $\Sigma(yZ - zY)$, $\Sigma(zX - xZ)$, $\Sigma(xY - yX)$, where

$$\Sigma X = X_1 + X_2 + X_3 + \text{etc.} \quad (1)$$
$$\Sigma(yZ - zY) = (y_1Z_1 - z_1Y_1) + (y_2Z_2 - z_2Y_2) + \text{etc.} \quad (2)$$

with similar expressions for the other forces and couples. The resultant of the three forces through O is a single force R, given by the equation

$$R = \sqrt{\{(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2\}} \quad (3)$$

and acting along a line through O whose direction-cosines are $\frac{X}{R}$, $\frac{Y}{R}$, and $\frac{Z}{R}$. Now R cannot be zero unless the three following equations are simultaneously true:—

$$\Sigma X = 0 \quad (4)$$
$$\Sigma Y = 0 \quad (5)$$
$$\Sigma Z = 0 \quad (6)$$

Similarly the component couples about the three axes (whose moments we shall denote by L, M, and N for brevity) are equivalent to a single couple whose moment is given by

$$G = \sqrt{(L^2 + M^2 + N^2)} \quad (7)$$

The direction-cosines of the axis of this couple are $\frac{L}{G}$, $\frac{M}{G}$, and $\frac{N}{G}$. Just as R cannot be zero unless each of its components is zero, so G cannot be zero unless each of its components is zero, that is, unless

$$\Sigma(yZ - zY) = 0 \quad (8)$$
$$\Sigma(zX - xZ) = 0 \quad (9)$$
$$\Sigma(xY - yX) = 0 \quad (10)$$

But for equilibrium it is clearly necessary that both the force and the couple should be zero. The equations (4), (5), (6) and (8), (9), (10) are therefore the conditions of equilibrium of a rigid body acted on by any system of forces.

83. If all the forces pass through one point (x, y, z), equations (8), (9), and (10) are superfluous, because they follow from (4), (5), and (6). For, since x, y, and z are the same for every force

$$\begin{aligned}\Sigma(yZ - zY) &= y\Sigma Z - z\Sigma Y \\ &= 0\end{aligned}$$

because $\Sigma Z = 0$ and $\Sigma Y = 0$. Similarly (9) and (10) follow in the same way. Thus only three conditions are necessary for the equilibrium of forces through one point.

84. Parallel Forces acting at Points fixed in a Rigid Body.—If a system of parallel forces F_1, F_2, F_3, etc., act at points fixed in a rigid body, we will show that, whatever be the direction of the forces, the resultant always passes through another point fixed in the body. This point is called the *centre* of the parallel forces. This theorem has already been proved for forces in one plane (Art. 75). We shall now prove it for forces in three dimensions.

Let the co-ordinate axes be fixed in the body.

Let R denote the resultant of the forces, and let $\bar{x}$, $\bar{y}$, $\bar{z}$ be a point on its line of action. Now, since the resultant of a pair of parallel forces is a parallel force whose magnitude is the algebraic sum of their magnitudes, and since any number of parallel forces can be added two at a time, it follows that the resultant of any number of parallel forces is a force parallel to the given forces, and its magnitude is the algebraic sum of their magnitudes. Hence

$$\begin{aligned} R &= F_1 + F_2 + F_3 + \dots \\ &= \Sigma F \end{aligned}$$

If some of the forces F_1, F_2, F_3, etc., act in the opposite direction to the others, they must, of course, be considered as negative forces.

Now $-R$ acting through $(\bar{x}, \bar{y}, \bar{z})$ is in equilibrium with the given forces. Also $X_1 = lF_1$, $Y_1 = mF_1$, $Z_1 = nF_1$, where l, m, n, are the direction-cosines of F_1. Therefore, equations (8), (9), and (10), applied to $-R$ and the given system of forces, give

$$\begin{aligned} \Sigma(ynF - zmF) - \bar{y}nR + \bar{z}mR &= 0 \\ \Sigma(zlF - xnF) - \bar{z}lR + \bar{x}nR &= 0 \\ \Sigma(xmF - ylF) - \bar{x}mR + \bar{y}lR &= 0 \end{aligned}$$

Since l, m, and n, are the same for every force we may write the above equations thus:—

$$n(\Sigma yF - \bar{y}R) - m(\Sigma zF - \bar{z}R) = 0 \quad \dots \quad (11)$$
$$l(\Sigma zF - \bar{z}R) - n(\Sigma xF - \bar{x}R) = 0 \quad \dots \quad (12)$$
$$m(\Sigma xF - \bar{x}R) - l(\Sigma yF - \bar{y}R) = 0 \quad \dots \quad (13)$$

Now these equations are clearly satisfied for all values of l, m, and n, provided

$$\begin{aligned} \Sigma xF - \bar{x}R &= 0 \\ \Sigma yF - \bar{y}R &= 0 \\ \Sigma zF - \bar{z}R &= 0 \end{aligned}$$

that is, provided

$$\bar{x} = \frac{\Sigma xF}{R} = \frac{\Sigma xF}{\Sigma F} \quad \dots\dots \quad (14)$$

$$\bar{y} = \frac{\Sigma yF}{\Sigma F} \quad \dots\dots \quad (15)$$

$$\bar{z} = \frac{\Sigma zF}{\Sigma F} \quad \dots\dots \quad (16)$$

The meaning of this result is that, whatever be the direction of the given forces, if their points of application are fixed in the body and their magnitudes are constant, then the resultant passes through the point $(\bar{x}, \bar{y}, \bar{z})$, which is also fixed in the body.

The three equations (11), (12), (13) are not independent. There are only two independent equations, which may be written in the form

$$\frac{\bar{x} - \frac{\Sigma x\mathrm{F}}{\mathrm{R}}}{l} = \frac{\bar{y} - \frac{\Sigma y\mathrm{F}}{\mathrm{R}}}{m} = \frac{\bar{z} - \frac{\Sigma z\mathrm{F}}{\mathrm{R}}}{n}$$

These equations do not determine $\bar{x}, \bar{y}$, and $\bar{z}$, but they show that, for all values of l, m, and n, the point $(\bar{x}, \bar{y}, \bar{z})$ lies on a line through the fixed point whose co-ordinates are $\frac{\Sigma x\mathrm{F}}{\mathrm{R}}, \frac{\Sigma y\mathrm{F}}{\mathrm{R}}, \frac{\Sigma z\mathrm{F}}{\mathrm{R}}$. This is, of course, only what we should expect, for in seeking $\bar{x}, \bar{y}$, and $\bar{z}$, we are seeking a point on the line of action of R, and our equations naturally give us all points on that line. But our equations tell us, in addition, that all the lines of action for different directions of the given forces pass through the fixed point mentioned above.

85. Gravitation.—It was pointed out in Art. 46 that a body is pulled towards the earth's surface by a force which is called its weight. The weight of a body near the earth's surface is an attraction which the earth exerts on the body. Since the earth attracts all bodies near its surface it seems natural that it should attract distant bodies as well. Newton discovered and enunciated the law of universal gravitation, and this law has never seriously been questioned since his day. The law states that two particles of masses m_1 and m_2, at a distance r apart, attract each other with a force whose magnitude is $\kappa\frac{m_1 m_2}{r^2}$, where κ is a constant, which is the same for all particles and all distances, and it is called the *constant of gravitation.*

The above law, it must be observed, is only true for particles, but it will be proved, in the chapter on attractions, that a spherical body, whose density at any point depends only on the distance of that point from the centre, attracts external bodies with exactly the same force as if the sphere were concentrated into a particle of equal mass at its centre. Now Kepler had proved from observations that the planets describe ellipses with the sun in one focus, and from this fact alone it follows that the sun's attraction on any one planet varies inversely as r^2. Newton applied his theory to the motion of the moon, and he found that this motion agrees perfectly with his law of gravitation. Again, another of Kepler's inferences from observation of the motions of the planets leads to the conclusion that the sun's attraction on any planet acts in the line joining the planet to the sun's centre. These questions will be treated again in the chapters on dynamics of a particle.

All bodies attract all other bodies. The weight of a body is simply a particular instance of this general law. All bodies at the earth's surface are attracted towards the earth's centre with forces which vary inversely as the squares of their distances from its centre. The ratio of

the force at x feet above the surface to that at the surface on the same body

$$= \frac{R^2}{(R+x)^2}$$

where $R = 4000 \times 5280$, the earth's radius in feet.

It is obvious that for all ordinary values of x, *i.e.* for bodies near the earth's surface, this variation with distance is extremely small. For this reason we may assume, in most cases, that the weight of a body is constant.

Again, the different particles of the same body at the earth's surface are pulled towards the earth's centre. But for any body of ordinary magnitude the lines joining different particles to the earth's centre are so nearly parallel that we may assume they are parallel. This, then, is our assumption: that the weights of the different parts of a body act along parallel lines, and the weight of each portion is constant. Moreover, whenever necessary, we assume that the weights of bodies are proportional to their masses because the attracting mass is the same for every body.

86. Centre of Gravity.—The weight of a body is the sum of the weights of the particles of the body, which weights are a system of parallel forces. We shall show that the weight of a rigid body always acts through a point fixed in the body however it may be turned relatively to the earth. This fixed point is called the *Centre of Gravity* of the body. It is sometimes called the *Centre of Mass*. This latter name occurs frequently in dynamics, where it is proved that the point has important properties which are in no way related to gravitation.

We have now to prove that the weight of a rigid body does always act through a fixed point.

The weight of a rigid body is the resultant of the weights of the particles of the body, these weights being a system of parallel forces acting at fixed points in the body. Now as the body is turned relatively to the earth the weights turn relatively to the body. The weights are therefore such a system of forces as we have dealt with in Art. 84, and we showed there that the resultant always acts through a point fixed in the body. If axes OX, OY, OZ, be taken in the body and w_1, w_2, w_3, etc., be the weights of the particles situated at $(x_1y_1z_1)$, $(x_2y_2z_2)$, $(x_3y_3z_3)$, etc., the co-ordinates of the centre of gravity are $\bar{x}$, $\bar{y}$, $\bar{z}$, given by the equations

$$\left.\begin{aligned} \bar{x} &= \frac{w_1x_1 + w_2x_2 + w_3x_3 + \dots}{w_1 + w_2 + w_3 + \dots} = \frac{\Sigma wx}{\Sigma w} \\ \bar{y} &= \frac{\Sigma wy}{\Sigma w} \\ \bar{z} &= \frac{\Sigma wz}{\Sigma w} \end{aligned}\right\} \quad . \quad . \quad (17)$$

87. Suppose we want to find the centre of gravity of a body composed of several finite bodies, rigidly connected, of weights w_1, w_2, w_3, etc., whose centres of gravity are known to be at $(x_1y_1z_1)$, $(x_2y_2z_2)$,

$(x_3y_3z_3)$, etc. Just as in dealing with particles, we are finding the resultant of a system of parallel forces acting at fixed points in a rigid body. The centre of gravity is found by exactly the same formulæ as if the finite bodies were particles each situated at its centre of gravity Thus equations (17) will give the co-ordinates of the centre of gravity of the compound body.

88. Continuous Bodies.—The formulæ (17) would not be of much use for ordinary rigid bodies, for even though a rigid body is composed of molecules it would be a severe task to find the centre of gravity by taking account of every molecule. A rigid body is therefore treated as a *continuous* body. That is, we suppose that the mass in a small volume surrounding any point in the body varies continuously with the volume. To put it in symbols, we assume that, if a small volume δv contains a mass δm, the fraction $\frac{\delta m}{\delta v}$ approaches a finite limit as δv is made infinitely small. The limiting value of this fraction is called the density, and it will be denoted by ρ. It follows from this that if m denotes the mass of any body

$$\frac{dm}{dv} = \rho$$

and therefore $$\int dm = \int \rho dv$$

the integral embracing every element of volume in the body.

Also in the formulæ for the co-ordinates of the centre of gravity we can put gdm for one of the w's. Thus

$$\bar{x} = \frac{\int x \,.\, gdm}{\int gdm} = \frac{\int xdm}{m}$$

But $$dm = \rho dv.$$

Hence $$\bar{x} = \frac{\int x\rho dv}{m}$$

If ρ is constant throughout the body, and V denotes the total volume, we get

$$\bar{x} = \frac{\int xdv}{V}$$

with two similar expressions for $\bar{y}$ and $\bar{z}$.

When we speak of the centre of gravity of an area, it is to be understood that the area is regarded as an infinitely thin uniform plate.

89. The centres of gravity of many bodies are known from considerations of symmetry. The following are some examples. The density is supposed to be constant unless the contrary is stated.

(1) The centre of gravity of a thin uniform rod is at its mid-point.

(2) The centre of gravity of a thin uniform plate bounded by two concentric circles is at the common centre of the circles. This includes the cases of a complete circular plate (the inner radius being zero) and a circular hoop or wire (the inner radius being nearly equal to the outer).

(3) The centre of gravity of a solid, bounded by two co-axial

cylinders and planes perpendicular to the axis, is at the mid-point of the axis. If the inner radius is zero we get the case of a complete cylinder.

(4) The centre of gravity of a sphere is at its centre. This is true if the density is constant, or if it is a function of the distance from the centre.

(5) The centre of gravity of a uniform rectangular plate is at the intersection of its diagonals.

90. Centre of Gravity of a Parallelogram.—Although it does follow from symmetry that the centre of gravity of a uniform thin parallelogram is at the intersection of its diagonals, it is not so obvious as the preceding examples. We will therefore prove it.

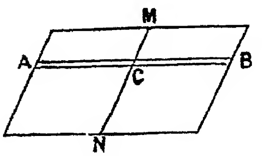


FIG. 29.

Let the parallelogram be divided into thin equal strips parallel to one pair of sides. Let AB be one of these strips, and let C be its mid-point. The strip AB is so thin that it may be regarded as a uniform thin rod, and its centre of gravity is therefore at its mid-point C. Now the centre of gravity of all strips such as AB are on the line MN bisecting the sides to which the strips are parallel. Thus the weight of the parallelogram can be replaced by a series of equal forces—the weights of the equal strips—at equal distances along MN. The "centre" of these parallel forces must clearly lie in the line MN, because it is a fixed point for all directions of the forces, and if we take the forces along MN the resultant must act along the same line. Also, since the forces are equal and uniformly distributed along MN, the centre of the parallel forces, which is the centre of gravity of the parallelogram, is at the mid-point of MN.

We might have regarded the weights of the strips acting at equidistant points in MN as equivalent to a uniform rod along MN, the centre of gravity of which we know by symmetry to be at its mid-point.

91. Centre of Gravity of a Uniform Triangular Plane.—Let the triangle be divided into infinitely thin strips, such as PQ, parallel to one of the sides. The weight of PQ can be replaced by the weight of a particle of equal mass at S, the mid-point of PQ. Hence the weight of the whole triangle can be replaced by the weights of a series of particles distributed along AM, which bisects all strips parallel to BC. But these weights are not equivalent to a uniform rod, because if all the strips have equal widths the particles are at equal distances apart, but they increase in weight from A to M. Nevertheless the resultant weight does act somewhere in the line AM. Again by taking strips parallel to the side AC it follows that the weight acts through a point in the line joining B to N, the mid-point of AC. The centre of gravity is therefore at G, the intersection of AM and BN.

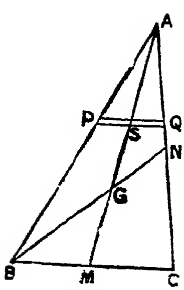


FIG. 30.

The third median of the triangle, the line joining C to the mid-point of AB, passes through G by a well-known geometrical theorem. Also by easy geometry it is proved that MG $= \frac{1}{3}$AM, and similarly each of the other medians is trisected at G.

92. The centre of gravity of a triangular plate is situated at the same point as three equal particles placed either

(1) at its angular points,

or (2) at the mid-points of its sides.

We will prove (1), and leave (2) to be proved by the student.

Suppose three particles each of weight w are placed at A, B, and C (Fig. 30). The two weights w at B and C are equivalent to a weight $2w$ at M. Now a force $2w$ at M and a parallel force w at A have a resultant $3w$ at a point G′ in AM, such that 2MG′ = G′A. Thus G′ is the same point as G, and it is the centre of gravity of the three particles at A, B, and C.

93. Centre of Gravity of a Uniform Rectilinear Plate.—The simplest method of finding the centre of gravity of a uniform plate having the shape of a polygon is to divide the polygon into triangles, and replace each triangle by three particles at its corners, each particle having a mass equal to one-third of the mass of its triangle, and then to find the centre of gravity of this system of particles. We will apply this method to find the centre of gravity of a quadrilateral.

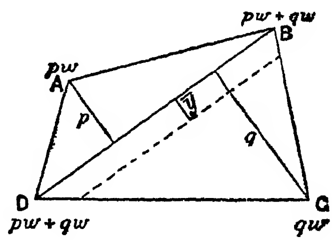


FIG. 31.

Let ABCD be the quadrilateral divided into two triangles by the diagonal BD. The perpendiculars from A and C on this diagonal are p and q. Now the areas of the triangles, and therefore their weights, are proportional to p and q. Let us therefore take the weights of ABD and CBD to be $3pw$ and $3qw$. Then the triangles are replaced by particles at the corners so that there is pw at A, qw at C, and $(p+q)w$ at each of the corners B and D. Then, taking DB as axis of x, the positive direction of the axis of y towards C, the formulæ

$$\bar{y} = \frac{\Sigma wy}{\Sigma w}$$

gives in this case

$$\bar{y} = \frac{q^2w - p^2w}{3(p+q)w}$$

$$= \tfrac{1}{3}(q-p)$$

Thus the centre of gravity lies on a line parallel to BD and at a distance $\frac{1}{3}(q-p)$ from it on the side towards C. Similarly the distance from the other diagonal can be expressed in terms of the perpendiculars from B and D on this diagonal. Thus the position of the centre of gravity is determined.

94. Centre of Gravity of a Solid Pyramid of Uniform Density.—Suppose the pyramid to be divided into infinitely thin slices parallel to the base. Each of these slices is similar to the base, and the centre of gravity of each lies on the line joining the vertex V to the centre of gravity C of the base. The weight of each slice can be replaced by the weight of a particle of equal mass at its centre of gravity. We therefore have to find the centre of gravity of a series of particles situated in VC whose weights are the same as those of the slices.

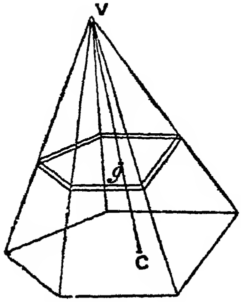


FIG. 32.

Let g be the centre of gravity of one slice, and let $Vg = x$, $VC = l$. If p is the perpendicular from V on the upper plane of the slice, and $p + dp$ on the lower plane,

$$\frac{p}{x} = \text{a constant for every slice}$$

$$= k \text{ say;}$$

therefore $$dp = kdx$$

Now the area of the slice is proportional to x^2, and consequently its weight is proportional to x^2dp, that is, proportional to x^2dx. Suppose the weight is wx^2dx. Then

$$\bar{x} = \frac{w\int_0^l x^3dx}{w\int_0^l x^2dx}$$

$$= \tfrac{3}{4}l$$

Thus the centre of gravity, G, lies in VC in such a position that $VG = \frac{3}{4}VC$.

95. Centre of Gravity of a Cone.—The case of a solid cone need not be treated separately. It is included in the pyramids. The method used in the last article is applicable to all pyramids on rectilinear or curvilinear bases. Thus the centre of gravity of a right circular cone is on the axis at three-quarters the distance from the vertex to the base.

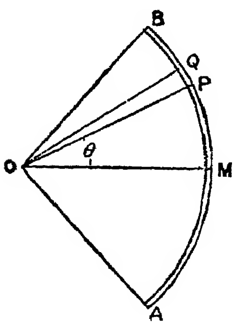


FIG. 33.

96. Centre of Gravity of a Uniform Rod in the Form of a Circular Arc.—Let AMB be the arc, and M its mid-point. Let r be the radius, and O the centre of the circle of which AMB is a part. The angle subtended at O by AMB is denoted by 2α.

Now the centre of gravity lies on OM from symmetry. Let G be the centre of gravity, P any point on the arc MA, Q a neighbouring point. $MOP = \theta$, $MOQ = \theta + d\theta$. If w

denotes the weight of unit length of the rod, the weight of PQ is $wrd\theta$. Then taking OM as axis of x,

$$OG = \bar{x} = \frac{\int_{-\alpha}^{+\alpha} wrd\theta \, . \, r\cos\theta}{\int_{-\alpha}^{+\alpha} wrd\theta}$$

$$= \frac{2r^2 \sin\alpha}{2r\alpha} = r\frac{\sin\alpha}{\alpha}$$

where α in the denominator denotes the number of radians in the angle.

97. Centre of Gravity of a Circular Sector.—The problem is to find the centre of gravity of a uniform plate in the form of a circular sector. Let the sector be divided into sectors such as OPQ, so small that they may be considered as triangles. The area of OPQ is $\frac{1}{2}r^2d\theta$, and the distance of its centre of gravity g from O is $\frac{2}{3}r$. Hence, if G denotes the centre of gravity of the sector, taking area for weight,

$$OG = \frac{\int_{-\alpha}^{+\alpha} \frac{1}{2}r^2d\theta \, . \, \frac{2}{3}r\cos\theta}{\int_{-\alpha}^{+\alpha} \frac{1}{2}r^2d\theta}$$

$$= \frac{2}{3}r\frac{\sin\alpha}{\alpha}$$

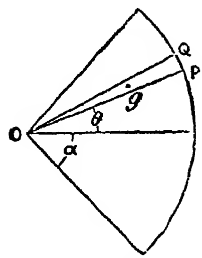


FIG. 34.

98. Centre of Gravity of a Zone of a Spherical Shell.—We shall now find the centre of gravity of a portion of a spherical shell of uniform thickness bounded by two parallel planes.

Suppose the zone is generated by the revolution of the arc AB about the diameter MOX. Let P and Q be two neighbouring points on this arc, and let YOP $= \theta$, YOQ $= \theta + d\theta$.

$$\left.\begin{array}{r}\text{The area gene-}\\ \text{rated by PQ}\end{array}\right\} = 2\pi \, . \, PQ \, . \, Pp$$

$$= 2\pi r^2 \cos\theta d\theta$$

Let O$p = x$, O$q = x + dx$. Then

$$x = r\sin\theta$$
$$dx = r\cos\theta \, d\theta$$

Hence the area generated by PQ

$$= 2\pi r dx$$

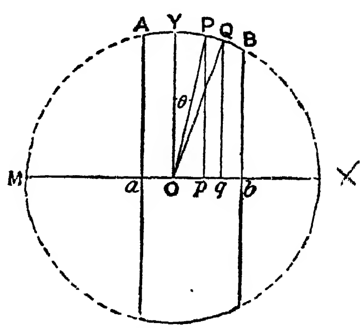


FIG. 35.

If G is the centre of gravity of the zone, and if the abscissæ of a and b are x_1 and x_2, then

$$OG = \frac{\int_{x_1}^{x_2} 2\pi r x dx}{\int_{x_1}^{x_2} 2\pi r dx} = \frac{\frac{1}{2}(x_2^2 - x_1^2)}{x_2 - x_1}$$
$$= \tfrac{1}{2}(x_2 + x_1)$$

Thus G is at the mid-point of ab, for it clearly lies on the line MX. This result applies to every size of zone, including a whole sphere.

It follows from the preceding investigation that the weight of a zone of a sphere acts at the same point as a rod along its symmetrical axis and bounded by the same planes.

99. Centre of Gravity of any Solid of Revolution of Uniform Density.—Suppose the surface of the solid is generated by the revolution about the axis of x, of the curve whose equation is

$$y = f(x)$$

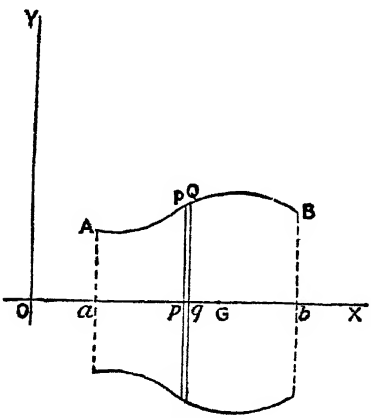


FIG. 36.

P and Q are two neighbouring points on the generating curve, p and q their projections on the axis of x. Let $Op = x$, $Oq = x + dx$. Then PQqp generates a circular disc whose volume is

$$\pi(pP)^2 . pq = \pi y^2 dx$$

Hence, since G is clearly on OX,

$$OG = \frac{\int \pi y^2 dx . x}{\int \pi y^2 dx}$$
$$= \frac{\int [f(x)]^2 x dx}{\int [f(x)]^2 dx}$$

The limits for x in the integrals are the values of x at the planes bounding the solid whose centre of gravity is required, that is, the values of x at a and b. The formula can only be used, of course, for solids bounded by planes perpendicular to the axis.

EXAMPLES.—(*a*) *Right circular cone.* The equation of the generating curve is

$$y = kx$$

the origin being at the vertex. Hence

$$OG = \frac{\int_0^h k^2 x^3 dx}{\int_0^h k^2 x^2 dx} = \frac{\frac{1}{4}h^4}{\frac{1}{3}h^3} = \tfrac{3}{4}h$$

which agrees with the result for a pyramid (Art. 94), of which a cone is a particular case.

(*b*) *Hemisphere.* The equation of the generating curve is now

$$y^2 = a^2 - x^2$$

Hence
$$\mathrm{OG} = \frac{\int_0^a (a^2 - x^2)x\,dx}{\int_0^a (a^2 - x^2)dx} = \frac{\frac{1}{2}a^4 - \frac{1}{4}a^4}{a^3 - \frac{1}{3}a^3}$$
$$= \tfrac{3}{8}a$$

(*c*) *Paraboloid of revolution.* Here the equation of the generating curve is

$$y^2 = kx$$

Therefore, if the solid is bounded by the planes $x = a$, $x = b$,

$$\mathrm{OG} = \frac{\int_a^b kx\,.\,x\,dx}{\int_a^b kx\,dx} = \frac{\frac{1}{3}(b^3 - a^3)}{\frac{1}{2}(b^2 - a^2)}$$
$$= \frac{2}{3}\cdot\frac{b^3 - a^3}{b^2 - a^2}$$

100. Centre of Gravity of a Hollow Body with a Symmetrical Axis.—We can suppose such a body obtained by rotating the area between two curves about the axis of x.

If AB and A′B′ are the two curves whose equations are

$$y = f(x)$$
$$y' = \mathrm{F}(x)$$

the volume generated by rotating a strip PQQ′P′ about OX is

$$(\pi y^2 - \pi y'^2)dx$$
$$= \pi[\{f(x)\}^2 - \{\mathrm{F}(x)\}^2]dx$$

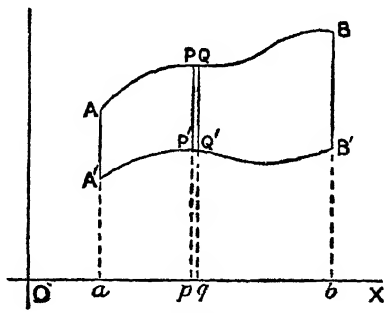


FIG. 37.

Now the weights of the strips are proportional to their volumes, and, since the density occurs in both numerator and denominator, we may take volume for weight. Hence, since G is on OX,

$$\mathrm{OG} = \frac{\int[\{f(x)\}^2 - \{\mathrm{F}(x)\}^2]x\,dx}{\int[\{f(x)\}^2 - \{\mathrm{F}(x)\}^2]dx}$$

We should, of course, get the same result by regarding the hollow body as composed of two complete solids of revolution, one generated by the curve AB with a positive weight, and the other generated by the

curve A′B′ with a negative weight. Then, if w_1 and $-w_2$ denote their weights, and x_1, x_2, the abscissæ of their centres of gravity, we should get

$$\mathrm{OG} = \frac{w_1x_1 - w_2x_2}{w_1 - w_2}$$

which will be found to agree with the previous result if w_1x_1 and w_2x_2 are calculated by the formula of the last article.

EXAMPLE.—*A solid is formed by cutting a right circular cone with height* h *and radius of base* r *from a cylinder with the same height and same base. To find the centre of gravity of the solid.*

Taking the origin at the vertex of the cone, the equations of the generating curves of the cylinder and cone are

$$y = r \quad \text{and} \quad y = \frac{r}{h}x$$

Hence

$$\mathrm{OG} = \frac{\int_0^h \left(r^2 - \frac{r^2}{h^2}x^2\right)x\,dx}{\int_0^h \left(r^2 - \frac{r^2}{h^2}x^2\right)dx} = \tfrac{3}{8}h$$

The following method may also be used to these questions. The cylinder may be regarded as the sum of the cone and the body whose weight is required. Hence the weight of the cylinder is the resultant of the weights of the two component bodies. If the weight of the cone be denoted by W, the weight of the cylinder is 3W, and the weight of the given body is 2W. We know that the centres of gravity of the cone and the cylinder are at $\frac{3}{4}h$ and $\frac{1}{2}h$ from O. Hence, by equating the moments about O of the cone and the hollow body to that of the cylinder, we get

$$\tfrac{3}{4}h\mathrm{W} + \mathrm{OG}\,.\,2\mathrm{W} = \tfrac{1}{2}h\,.\,3\mathrm{W}$$

Therefore

$$\mathrm{OG} = \tfrac{3}{8}h$$

It will generally be found safer and simpler to use this last method rather than the formula. It will only be necessary to find the volumes and centres of gravity of the hollow body and the body which would fill the hollow, and these are complete solids of revolution.

101. Centre of Gravity of a Solid of Variable Density.—Let ρ be the density at any point x, y, z. Then the mass enclosed by the parallelopiped whose edges are dx, dy, dz is $\rho dxdydz$ if the axes are rectangular. Hence

$$\bar{x} = \frac{\iiint \rho x\,dxdydz}{\iiint \rho\,dxdydz}$$

There are two similar expressions for $\bar{y}$ and $\bar{z}$.

102. Theorems of Pappus or Guldinus.

(1) If a plane curve be rotated about any line in its plane, the area of the surface thus traced out by the curve is equal to the product of

the length of the curve and the distance traversed by the centre of gravity of the curve, regarded as a rod of uniform density.

(2) If a plane area be rotated about any line in its plane, the volume described by the area in its rotation is equal to the product of the area and the distance traversed by the centre of gravity of the area.

These are the two theorems which we will now prove. It should be pointed out that, unless we introduce negative areas and negative volumes, the curve in (1) and the area in (2) should lie entirely on one side of the axis of rotation.

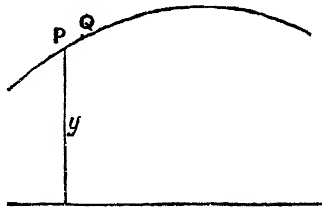


FIG. 38.

To prove (1).

Let PQ be a small element of the curve of length ds, and let the distance of P from the axis be y. Suppose the plane of the curve is rotated through an angle θ radians. Then the area traced out by PQ $= \theta y \, . \, ds$. Hence the area traced out by the whole curve

$$= \int \theta y ds = \theta \int y ds$$

But if l denotes the whole length of the curve and $\bar{y}$ the distance of its centre of gravity from the axis

$$\bar{y}l = \int y ds$$

Hence the area traced out $= \theta \bar{y} l$

Now $\theta \bar{y}$ is the distance traversed by the centre of gravity of the curve. Hence (1) is proved.

To prove (2).

Let A denote the area rotated, $\bar{y}$ the distance of its centre of gravity from the axis, dA an infinitely small element of the area, and y the distance of this element from the axis. Then the volume described

$$= \int \theta y d\mathrm{A} = \theta \int y d\mathrm{A}$$
$$= \theta \bar{y} \mathrm{A}$$

Thus (2) is proved.

103. We will give a few instances of the preceding theorems. They are useful in helping to recall the positions of the centres of gravity of certain bodies.

(*a*) Assuming we know the area of a sphere and the length of a semicircular arc, to find the centre of gravity of the arc.

Suppose the arc is rotated about its bounding diameter so as to describe a hemisphere. Let r be the radius. Then by Pappus' first theorem

$$\pi \bar{y} \, . \, \pi r = 2\pi r^2$$

Therefore $$\bar{y} = \frac{2}{\pi} r$$

(*b*) From the volume of a hemisphere and the area of a semicircle, to find the centre of gravity of the semicircle.

If the semicircle is rotated through two right angles it will describe a solid hemisphere. Therefore

$$\pi\bar{y}(\tfrac{1}{2}\pi r^2) = \tfrac{2}{3}\pi r^3$$

whence
$$\bar{y} = \frac{4}{3\pi}r$$

(*c*) To find the volume of an anchor ring, that is, the solid described by rotating a circle about a line in its plane through a complete revolution. By Pappus' second theorem the volume

$$= 2\pi b\,.\,\pi r^2$$

where r is the radius of the circle, and b is the distance of the centre of the circle from the axis of rotation.

(*d*) From the area and the position of the centre of gravity of a triangle, to find the volume of a cone.

A right-angled triangle of height r and base h, when rotated about its base, will describe a cone with height h and radius of base r. The centre of gravity of the triangle is at a distance $\tfrac{1}{3}r$ from the base. Hence

$$\text{Volume of cone} = 2\pi(\tfrac{1}{3}r)(\tfrac{1}{2}hr)$$
$$= \tfrac{1}{3}\pi r^2 h$$

EXAMPLES ON CHAPTER III

1. A tripod is formed of three bars each of length l, attached by smooth hinges to a small block from which a weight W is suspended. If the feet of the tripod rest on the ground at the corners of an equilateral triangle of side a, find the thrust in each rod.

$$\left[\frac{lW}{\sqrt{(9l^2 - 3a^2)}}\right]$$

2. Three strings are attached to points A, B, C, on the ceiling of a room. They are brought together at the lower ends and attached to the same point D of a weight W. Suppose the vertical plane containing AD cuts BC in K. Show that the resultant of the tensions in DB and DC acts along DK, and thence prove that the tension in AD is

$$W\frac{\cos AKD}{\sin ADK}$$

3. The tops of the legs of a three-legged table are situated at the corners of an equilateral triangle ABC with sides 3 feet long. A weight W is placed on the table at a distance 15 inches from BC and 12 inches from CA. Find the additional pressure on each leg.

[The additional pressures on A, B, C, are 0·481W, 0·385W, 0·134W.]

4. A triangular board ABC is supported at its corners. If a weight W is placed at any point D on the board, show that the additional pressures on the supports at A, B, C, are proportional to the areas of the triangles DBC, DCA, DAB, respectively.

5. If a smooth sphere whose centre of gravity is not at its centre, rests in contact with any number of smooth surfaces, show that the line joining its centre to its centre of gravity must be vertical.

6. Three equal smooth spheres are lying in contact on a horizontal plane and are held together by a string. A cube of weight W is placed with one diagonal vertical so that its lower faces touch the spheres, and the cube is supported in this position by the spheres; show that the tension in the string is

$$\tfrac{1}{3}\sqrt{\tfrac{2}{3}}W$$

London B.Sc.

7. ABCD, ABEF, are two faces of a cube; there is a system of five forces represented respectively by AB, BC, CD, DA, and FE.

(*a*) Find their resultant.

(*b*) Show that the first four forces when compounded with any fifth force acting along FE will reduce to a single resultant.

In (*b*), if the direction of the first force is reversed, show that in one particular case a force can act along FE such that the five forces cannot be reduced to single resultant.

8. ABCD, A′B′C′D′, are a pair of opposite faces of a cube with sides *a* feet, and AA′, BB′, etc., are perpendicular to these faces. If equal forces of magnitude P act along DC, BB′, A′D′, show that they can be reduced to a force along AC′ and a couple whose vector is parallel to the force. Find the magnitudes of the force and the couple.

[$\sqrt{3}$P, $\sqrt{3}a$P.]

9. Show that a force at any point can be replaced by an equal and parallel force at any other point together with a couple whose vector is perpendicular to the plane of the two forces.

Thence show that, in a number of equivalent sets, each consisting of a force and a couple in space, the common parts of every set are, (1) the magnitude and direction of the resultant force, and (2) the component of the couple vector parallel to the force.

10. A force F acts vertically upwards through a point at the foot of a plane inclined at α to the horizontal. A couple M acts in the inclined plane, the direction of its vector being towards the upper side of the plane. Show that the force and couple are equivalent to a couple in a horizontal plane and an upward force F acting at a point in the foot of the inclined plane at distance $\frac{M\sin\alpha}{F}$ from the original point of application, the new point of application being to the right of the old one from the point of view of a man looking up the plane.

11. Reduce the forces in question 8 to a force at D and a couple. What is now the magnitude of the couple, and what is the direction of the vector representing it? Show that the component of this couple vector parallel to the resultant force is the same as in question 8.

[The couple is $\sqrt{5}a$ P parallel to the line joining A to the middle of BC.]

12. Let AB, AC, AD, be three edges of a cube, and consider AD and two other edges parallel to AB and AC respectively, which intersect neither AD nor one another. Suppose that equal forces P act along these edges in the directions AB, AC, AD. Show how to reduce them to a resultant force and a couple. Also show how to draw a line along which the resultant acts when the plane of the couple is at right angles to the direction of the resultant.

[Let AZ be a diagonal of the cube, YZ the edge along which the force through Z acts. Then the resultant force is parallel to AZ, and when the plane of the couple is perpendicular to this, the force must act at a point in OM distant $\frac{\sqrt{2}}{3}$ of an edge from AZ, O being the centre of the cube and M the mid-point of DY.]

13. A uniform rod BC is divided into two parts at A, which are joined by a hinge. If AB is fixed and AC moves in a plane round A, show that the centre of gravity of the divided rod describes a circle. Also show how to draw the circle when AC is one-third of the length of the rod.

[The centre of gravity describes a curve similar to the one described by C (a circle) with the mid-point of AB as centre of similitude.]

14. If any portion (of volume v) of a body or system of bodies (whole volume V) be displaced to another position, the displacement GG′ of the centre of gravity of the whole is parallel to gg' the displacement of the centre of gravity of the portion, and its amount is given by

$$GG' = \frac{v}{V} gg'$$

Prove this theorem.

15. Three particles, whose masses are P, Q, and R, are placed to the corners of a triangle; Q and R remain fixed, while P moves along the sides of the triangle successively: show that the centre of gravity of the three particles describes the sides of a triangle similar to the given triangle, and define the position of this triangle with respect to the given triangle.

[One side of the triangle lies along QR, the other two meet at G, the centre of gravity of P, Q, and R, before P is moved, and the triangle is similar to the given triangle.]

16. A uniform square plate of one foot side has two circular holes punched in it, one of radius one inch, co-ordinates of centre (4, 5) inches, referred to two adjacent sides of the plate as axes; the other of radius half an inch at the point (8, 1) inches: find the co-ordinates of the centre of mass of the remainder of the plate. *London Inter. Sci.*

[At $x = 6{\cdot}03$, $y = 6{\cdot}05$ approximately.]

17. A hollow conical vessel made of thin sheet metal is closed at its base; if it is cut across by a plane parallel to its base at half the perpendicular height, and the upper cone removed, prove that the distance from the base of the centre of gravity of the remainder of the vessel is

$$\frac{2}{3} \cdot \frac{lh}{3l + 4r}$$

where h = the perpendicular height, l = the length of the slant side, r = the radius of the base. *London Inter. Sci.*

18. A uniform solid hemisphere of weight W rests with its curved surface on a smooth horizontal plane. A body of weight P is suspended from the rim of the hemisphere: find the inclination of the base of the hemisphere to the horizontal in the position of equilibrium. *London B.Sc.*

$$\left[\text{Inclination} = \tan^{-1}\frac{8}{3} \cdot \frac{P}{W} \cdot\right]$$

19. Find the centre of gravity of a solid hemisphere whose density varies as the square of the distance from the centre of the whole sphere.

[At $\frac{5}{12}$ of the radius from the centre.]

20. Find the centre of gravity of a semicircular disc whose thickness varies as the square of the distance from the circumference.

[At $\frac{4}{5\pi}$ of the radius from the centre.]

·21. Find the centre of gravity of a solid hemisphere whose density varies as the square of the distance from the centre of the curved surface.
Manchester University.

[First show that the mass of a disc of thickness δx at distance x from the centre of the curved surface, is proportional to $x^2(4r^2 - x^2)\delta x$. The centre of gravity is $\frac{25}{34}r$ from the centre of the curved surface, r being the radius.]

22. Find the centre of mass of half an ellipsoid bounded by a principal plane perpendicular to a diameter of length $2b$.

The ellipsoid is cut by a plane parallel to this plane and the thickness of the slice is t; show that the centre of mass of the slice divides the thickness in the ratio of $6b^2 - t^2) : (6b^2 - 3t^2)$, where b is the semi-axis perpendicular to the planes of section.
Manchester University.

[Answer to the first part: at $\frac{3}{8}b$ from the centre.]

23. Find the centre of gravity of a wedge cut from a right circular cylinder by a pair of planes meeting in a tangent to a circular section of the cylinder.

[The centre of gravity is at the middle point of a line parallel to the axis of the cylinder, with its ends on the faces of the wedge, and at a distance $\frac{1}{4}$ of the radius from the axis.]

24. Find the centre of gravity of a portion of a right circular cylinder terminated obliquely by a plane.

Whatever be the inclination of the plane to the base, show that the centre of gravity can never be at a greater distance from the axis than a quarter of the radius.

[If L, l are the lengths of the longest and shortest generating lines on the cylinder, the centre of gravity lies at the mid-point of a line which is parallel to the axis and at distance $\frac{L - l}{L + l}\cdot\frac{r}{4}$ from it.]

25. A cylindrical vessel open at the top is filled with water and hung up by a point on its rim; at first the top is horizontal, but it is allowed to turn gradually round the fixed point, and it comes to rest when half the water has been poured out. Neglecting the weight of the vessel, show that the ratio of the height to the diameter of the base is $\sqrt{10} : \sqrt{11}$.

26. If G is the centre of gravity of any number of particles, m_1, m_2, m_3 etc., in space situated at P_1, P_2, P_3, etc., show that the vector sum

$$m_1\overrightarrow{P_1G} + m_2\overrightarrow{P_2G} + m_3\overrightarrow{P_3G} + \text{etc.}$$

is zero.

If Q is any other point, show that

$$m_1\overrightarrow{P_1Q} + m_2\overrightarrow{P_2Q} + m_3\overrightarrow{P_3Q} + \text{etc.} = (m_1 + m_2 + m_3 + \ . \ . \ .)\overrightarrow{GQ}.$$

[The equation

$$m_1x_1 + m_2x_2 + m_3x_3 + \ . \ . \ . = (m_1 + m_2 + m_3 + \ . \ . \ .)\bar{x}$$

is the x-component of the second of the given vector equations and the equation can be proved at once from the component-equations.]

CHAPTER IV

GRAPHICAL METHODS

104. Graphical Method of finding the Resultant of a Number of Coplanar Forces.—Let the given forces be P, Q, R, S, and T, acting as snown in Fig. 39A. Draw vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$, $\overrightarrow{DE}$, and $\overrightarrow{EF}$, to represent these five forces in the order named. Then we know that the resultant is represented by the vector $\overrightarrow{AF}$, and what we now require is its line of action.

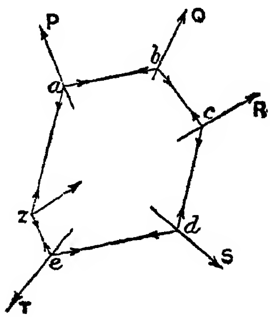


FIG. 39A.

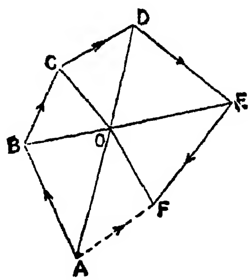


FIG. 39B.

Take any point O in the plane of the polygon of forces ABCDEF, and join O to the corners of the polygon.

Now the force P, represented by $\overrightarrow{AB}$, could be balanced by two forces represented by $\overrightarrow{BO}$ and $\overrightarrow{OA}$, if these acted through some point on its line of action. Take then any point *a* on the line of action of P, and draw through this point two lines *ab* and *az* parallel to BO and OA, the first of these lines meeting the line of action of Q in *b*. Again, through *b* draw a line *bc* parallel to CO and meeting the line of action of R in *c*. And similarly *cd, de, ez*, are drawn parallel to DO, EO, and FO.

Now let us imagine that *ab, bc, cd, de, ez, za*, are light rods, and that there are tensions (or thrusts if necessary) in these rods which are represented by BO, CO, DO, EO, FO, AO. Then, from the triangle AOB we see that the tensions in the rods *ab*, *az*, will

balance the force P; similarly, the forces Q, R, S, T, are balanced by the tensions in the rods meeting on their lines of action. But we are left with the tensions at the free ends of the rods *az*, *ez*, unbalanced. It is clear that a force at *z* which would balance the tensions there, that is, the force represented by $\overrightarrow{FA}$, would keep the whole system in equilibrium. And since the tensions in the rods, regarded as forces applied at *a*, *b*, etc., are a self-balancing system of forces, it follows that the force $\overrightarrow{AF}$ acting at *z* must be the resultant of the given forces.

Hence the polygon of forces ABCDEF gives the magnitude and direction of the resultant, and the polygon *abcdez*, which is called a *Funicular Polygon*, gives a point on the line of action of the resultant. Thus the resultant is completely determined.

The point O need not be inside the polygon of forces. We may take it outside the polygon if we choose. It is sometimes impossible to take O inside as will be seen in the following case.

105. Parallel Forces.—The method of finding graphically the resultant of a system of parallel forces is essentially the same as that for non-parallel forces. The only difference is that the polygon of forces is a *collapsed* polygon, all the corners lying on one straight line. We will work out an example.

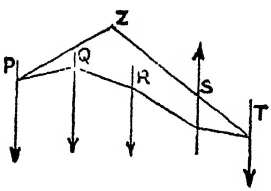


FIG. 40A.

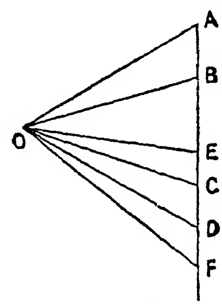


FIG. 40B.

Let it be required to find the resultant of P, Q, R, S, T. Vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$, $\overrightarrow{DE}$, $\overrightarrow{EF}$, are drawn to represent the forces in the order named. Then ABCDEF may be regarded as a closed polygon in which the last side FA happens to fall in the same line as the other five. There is then no difference between the method used here and the one given for intersecting forces. The resultant is $\overrightarrow{AF}$, and its line of action passes through Z.

106. Rigid Frameworks.—The methods of graphic statics can be applied to find the stresses in the rods of a framework which is just rigid.

A framework formed of rods, joined together by smooth hinges at

their ends, is rigid if its shape cannot be altered without altering the lengths of some of the rods. It is *just* rigid if the shape could be altered by removing any rod in the frame.

A framework is just rigid if the length of every rod is independent of the lengths of all the rest, while the distance between any pair of hinges, whether connected by a rod or not, depends on the lengths of some of the rods.

Except in particular cases where equal parallel rods are used, the number of rods in a framework which is just rigid is less than twice the number of hinges by three.

It does not follow that the frame will be rigid if this numerical relation between the number of rods and the number of hinges holds. The rods may be so disposed that some parts of the frame have more rods than are necessary for rigidity, while the remainder has less.

Let us consider how a rigid frame could be built up.

The simplest rigid frame is a triangle.

Any other rigid frame can be formed by joining one rigid frame (or a rod) to another rigid frame by a rigid connection.

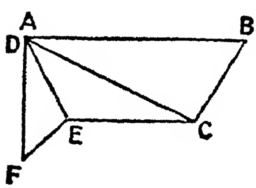


FIG. 41A.

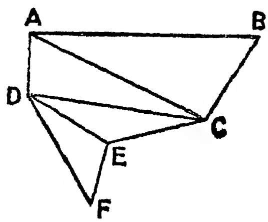


FIG. 41B.

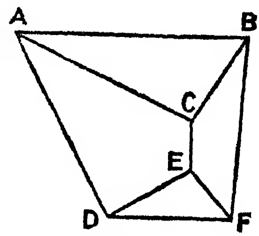


FIG. 41C.

There are two different ways of joining two rigid frames rigidly together. The first is by joining a corner of each to a common hinge, and then connecting a rod to one other hinge in each. The second method is to join them together by three rods, no two of which coincide (*i.e.* no two of which join the same hinges).

Thus the two triangles ABC, DEF, can be rigidly connected in the ways shown in Figs. 41A, 41B, 41C. Every one of these frames is just rigid, and in each case the numerical relation stated between the number of hinges and the number of rods is true. It is left to the student to prove that the relation is true for all frames built up by the methods indicated.

107. Over-rigid Frames.—A frame containing more rods than are necessary for rigidity is *over-rigid*, and any rod that could be removed without affecting the rigidity of the rest is a *redundant* rod. In Fig. 42A any rod in the frame is redundant, but in Fig. 42B any rod except BE and CE may be regarded as redundant.

There may be stresses in an over-rigid frame without the application of any external forces. For it is clear that, when a redundant rod

is introduced which does not exactly fit the position it is intended for, this rod will exert forces on the hinges to which it is joined, and these will cause stresses in the other members of the frame.

In dealing with frameworks joined together by smooth hinges and acted on by forces at the corners only, it is important to bear in mind that the action of each rod on its hinge is a force along the line of the rod itself. For, since there are only two forces on each rod, namely, the actions at its hinges, and since two forces can be in equilibrium only if

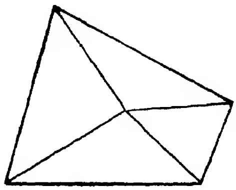


FIG. 42A.

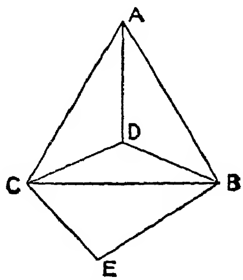


FIG. 42B.

they act in the same straight line, this straight line must be the line joining the two hinges. That is, the action of a hinge on any rod attached to it is a force along that rod, and the action of the rod on the hinge is that force reversed.

108. It is necessary to warn the student that the statement in the preceding paragraph is true for smooth hinges only. For, by casual observation, we learn that, when a rod is attached to a fixed hinge at one end, a considerable force may sometimes be applied at right angles to the rod at some distance from the hinge without affecting the equilibrium. The action of the hinge is not at a point, but is distributed over a cylinder. The resultant of all these forces acting over a cylinder may be regarded as a force through the centre of the hinge together with a couple, and it has been shown in Art. 71 that a force and a couple are equivalent to a force acting in another line.

109. The couple is formed by the friction at the hinge. The subject of friction will be treated in Chapter VII., but we may point out here how the couple arises. Fig. 43 represents the portion of the rod in the neighbourhood of the hinge of which A is the centre. When such a force as P is applied there is a frictional force called into play at every point of contact between the rod and the hinge. These frictional forces act in the surface of contact, and their directions on the rod are indicated

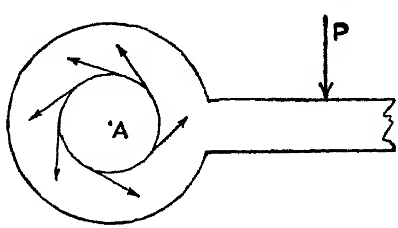


FIG. 43.

by the arrows in the figure. The forces perpendicular to the surface (which are not indicated in the figure) all pass through A, and therefore have a resultant through that point. If the moment of the frictional forces about A is sufficient to balance the moment of P, there can clearly be equilibrium.

It is obvious that the smaller the radius of the hinge (or axle) is, the less is the moment of given frictional forces about A, and consequently the less P would have to be for equilibrium. Thus, although there is always some friction at a hinge, if the hinge has a small radius and the forces applied to the frame are large, so that the couple at any hinge is small compared with the product of the smallest of the applied forces and the length of a rod, we are justified in treating the hinges as smooth. In this chapter we shall deal with smooth hinges only.

110. Stresses in a Framework acted on by Forces at the Hinges only.—The method will be shown by an example. We will find the stresses in the framework whose rods form the sides of the quadrilateral ABCD and the diagonal BD (Fig. 44A), when known forces P and Q are applied at A and B, a force R, whose direction only is known, is applied at C, and another force keeping the whole in equilibrium is applied at D. No other forces are supposed to act on the system.

Acting on the hinge at A are three forces, namely, P, and the thrusts (or tensions) of the rods AB, AD. Since the hinge is in equilibrium we can find the thrusts in the rods from a triangle of forces. We begin then by drawing this triangle of forces; it is the triangle named $\alpha\beta\gamma$ in Fig. 44B. On the same scale as $\overrightarrow{\alpha\beta}$ represents P, the vector $\overrightarrow{\gamma\alpha}$ represents the force which the rod AB exerts on the hinge at A. Consequently $\overrightarrow{\alpha\gamma}$ represents the action of the same rod on the hinge at B. We are now in a position to find the stresses in the rods BC, BD, by considering the equilibrium of the hinge at B. We draw $\overrightarrow{\delta\alpha}$ to represent the force Q. The vector for Q is put in this position because we want $\overrightarrow{\delta\gamma}$ to give the resultant of the two known forces at B, namely, the force Q and the thrust in the rod AB. We now draw $\gamma\epsilon$ and $\delta\epsilon$ parallel to the rods BD and BC. Thus $\delta\alpha\gamma\epsilon$ is a polygon of forces for the actions on the hinge at B; and since these forces are all to be taken from one corner to the next in order, the forces on the hinge at B are $\overrightarrow{\delta\alpha}$, $\overrightarrow{\alpha\gamma}$, $\overrightarrow{\gamma\epsilon}$, and $\overrightarrow{\epsilon\delta}$. These tell us that there is a tension in the rod BD and a thrust in BC, since the action of BD is away from the hinge and that of BC is towards the hinge. We next draw $\delta\zeta$ and $\epsilon\zeta$ parallel to the force R and the rod CD. The actions on the hinge at C being represented by the sides of the triangle $\delta\epsilon\zeta$ the magnitude of R is determined, and it is seen that there is a tension in the rod CD whose magnitude is represented by $\epsilon\zeta$. The actions on the hinge at D are represented by $\overrightarrow{\zeta\epsilon}$, $\overrightarrow{\epsilon\gamma}$, and $\overrightarrow{\gamma\beta}$. Hence, for the equilibrium of this hinge, there must be a force given by $\overrightarrow{\beta\zeta}$

applied to it. A dotted line is drawn in Fig. 44A to indicate this force. Thus the stresses in all the rods are determined as well as the elements of the external forces which were not given.

It will be seen that the essential part of the process consists in drawing a polygon of forces for the actions at each hinge, starting at a point where only two rods meet and a known force acts whenever this is possible. If no such point exists some preliminary work is necessary which will be indicated in Art. 114.

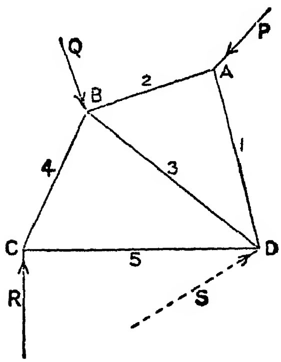


FIG. 44A.

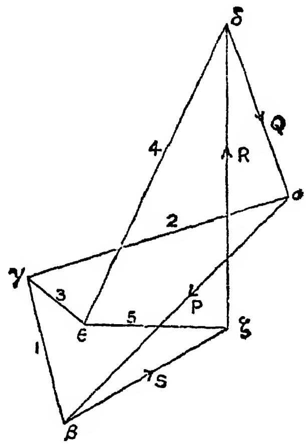


FIG. 44B.

111. Whenever a rigid framework at rest is acted on by a system of external forces, these forces must, of course, satisfy the general conditions of equilibrium. We could not therefore arbitrarily fix the forces at all the hinges; three elements must be left to be determined by the three conditions of equilibrium for coplanar forces. In the problem worked out the three elements which are left undetermined are (1) the magnitude of R, and (2) and (3) the two components of the force at D. Fig. 44B shows that the four external forces have no single resultant since they form the sides of the closed quadrilateral $\alpha\beta\zeta\delta$.

112. As we have pointed out before, there may be initial stresses in an over-rigid framework without the application of any external forces, owing to the imperfect adjustment of the lengths of the rods of the frame. If there is only one redundant rod in a frame we can find the ratios of the initial stresses in the rods by considering the thrust (or tension) in this redundant rod as two forces applied at the hinges to which it is attached. The problem is then exactly similar to the one we have just worked out. An example will make the process clear.

113. Determination of Initial Stresses.—The frame ABCD (Fig. 45A), with two diagonal rods AC, BD, is over-rigid, and has one redundant rod. Any rod in the frame can be considered redundant

since the frame would be rigid whichever were removed. Let us assume a thrust in one rod, say BD, and find the stresses in the remaining rods.

It is not necessary to give the details of the construction of the stress-diagram, Fig. 45B. The force marked with any number in the

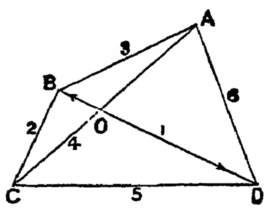


FIG. 45A.

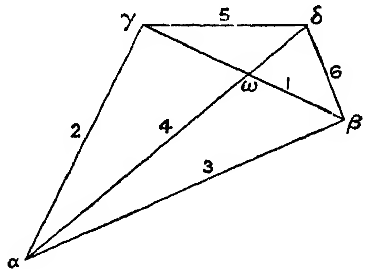


FIG. 45B.

stress-diagram is the stress in the rod with the same number in the frame. Also the lines in the stress-diagram are drawn in the orders of their numbers. We start at the corner B and then proceed to C. We have then found all the points α, β, γ, and δ, the stress 6 being the only one that has not been determined in the process. But we need no new points to find the actions on the hinge at D, for $\overrightarrow{\delta\gamma}$ and $\overrightarrow{\gamma\beta}$ are two of them; the remaining one must therefore be given by $\overrightarrow{\beta\delta}$.

It can easily be proved by geometry that $\delta\beta$ is parallel to AD. For, by similar triangles,

$$\frac{\mathrm{OA}}{\mathrm{OB}} = \frac{\omega\alpha}{\omega\beta}$$

$$\frac{\mathrm{OB}}{\mathrm{OC}} = \frac{\omega\gamma}{\omega\alpha}$$

$$\frac{\mathrm{OC}}{\mathrm{OD}} = \frac{\omega\delta}{\omega\gamma}$$

Multiplying corresponding sides of these three equations together, we get

$$\frac{\mathrm{OA}}{\mathrm{OD}} = \frac{\omega\delta}{\omega\beta}$$

Also by construction OA and OD are parallel respectively to $\omega\delta$ and $\omega\beta$. Hence the two triangles OAD and $\omega\delta\beta$ are similar, and their corresponding sides are parallel.

By consulting the stress-diagram and considering the forces at the different hinges, it will be seen that, if BD is in thrust, AC is also in thrust, and the other four rods are in tension. Also the equilibrium

would be undisturbed if all the tensions were changed to thrusts, and all the thrusts to tensions.

114. It often happens that more than two rods meet at the hinges where the known forces act. In such a case we have no simple starting-point for the stress diagram as we had in the two previous examples.

In such a case we have to do some preliminary work so as to determine an unknown force at a hinge where two rods meet, or else to determine the stress in one of the rods which is joined to only two others at a hinge.

These two methods apply to totally different cases. The first applies to the case of a frame which does contain a hinge at which only two rods meet but where an unknown force acts. The second applies to the case where two rigid frames are joined together by any of the methods explained in Art. 106.

The first case is very simple. We consider the equilibrium of the frame as a whole, and determine, either by the method of the funicular polygon or by analytical methods, the forces applied at all the hinges. Then since there is one corner where only two rods meet the procedure is the same as in Art. 110.

115. But when two rigid frames are jointed together there may be no hinge of the frame at which only two rods meet, as will be seen on examining Fig. 41C. Here, as in the preceding case, we had better begin by determining the forces applied at all the hinges by considering the equilibrium of the framework as one rigid body. Then, whether the two parts of the frame are joined together by a hinge and a rod, or by three rods, we can find the stresses in the rods connecting the two parts by considering the equilibrium of either part. Thus, if they are connected by a hinge and a rod, the stress in the rod is found by taking moments about the hinge of all the forces acting on one part (including the stress in the connecting rod) and equating their sum to zero. This will give the stress in the rod.

If the two parts are connected by three rods we can find the stress in any of the three by taking moments, about the intersection of the other two, of all the forces acting on one part and equating the sum to zero. This will give the unknown stress. Thus, if we choose, we can find the stresses in all the three rods in turn by this method.

The preceding rules are sufficient for any case where the stresses are determinate.

Two examples will now be worked showing the application of these rules.

116. ABCDE (Fig. 46A) is a given framework having known loads P and Q at B and C, and supported at A and D by vertical forces. The stresses in the members are required.

We cannot start at either of the corners B or C and determine the stresses in the rods meeting there, because three rods meet at each point and an infinite number of sets of forces along three concurrent lines can be found to balance any given force through their point of intersection. We are obliged, therefore, to start at A or D. But in order to do this we must first determine the force at the corner where we intend to begin.

Let us suppose that R has been determined either by the method given in Art. 104, or by taking moments about D for the equilibrium of the framework. Then the closed and collapsed polygon $\alpha\beta\eta\epsilon\alpha$ (Fig. 46B) is drawn to represent the forces on the framework. The triangle $\alpha\beta\gamma$ is formed to give the actions on the hinge at A. The actions at B are

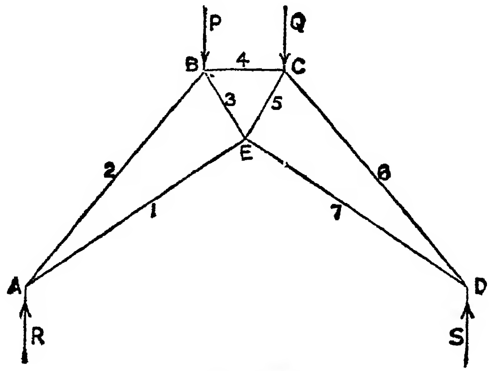


FIG. 46A.

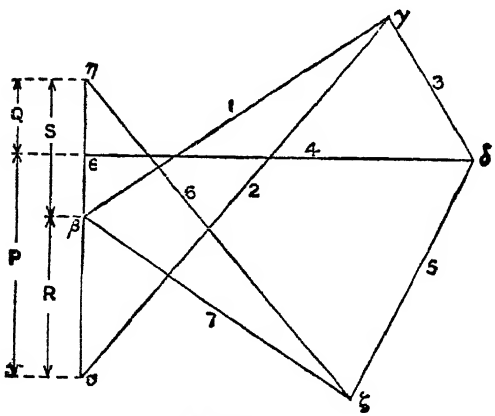


FIG. 46B.

now obtained by completing the quadrilateral $\epsilon\alpha\gamma\delta\epsilon$. Next the quadrilateral $\epsilon\delta\zeta\eta\epsilon$ is completed for the actions at C. Then the line $\zeta\beta$ represents the stress in DE. $\zeta\beta$ should therefore be parallel to DE, and the error in the direction of this line will give some idea of the accuracy of the working.

117. Let us now consider the frame in Fig. 47A. Here the central triangle DEF is rigidly connected to ABC by three rods, and the triangle GHK is rigidly connected to DEF in a similar manner. The rods BF and EH are supposed to cross without interfering with each

other. The three triangles so connected form one rigid frame. Equal loads W act at E and F, and the structure is supported by vertical forces W at A and K.

In this frame there is not a single hinge at which only two rods meet. We are obliged, therefore, to find the stress in one of the rods before we can begin to draw the stress-diagram.

The triangle GHK is a rigid body joined to another rigid body by three rods, and acted on by known forces. Consequently we can find the stress in any of the connecting rods by taking moments, about the

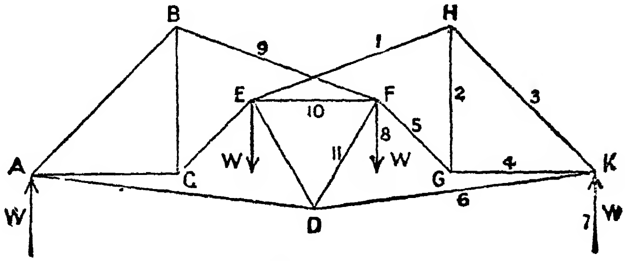


FIG. 47A.

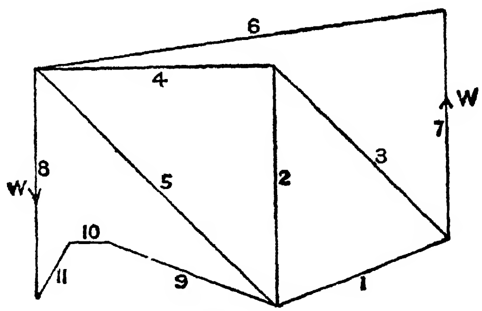


FIG. 47B.

intersection of the other two, of all the forces acting on the triangle, including the stress required, and equating the sum to zero.

By taking moments about the intersection of FG and DK we find that the stress in EH is a tension of magnitude about 0·79W. Now we can start at the hinge H and describe the stress-diagram shown in Fig. 47B. The same number is attached to a member of the frame and to the line in the stress-diagram representing the stress in that member. The numbers also indicate the order in which the stresses are obtained in the diagram. Thus 1, 2, 3, are obtained from the triangle for the hinge H; then 2, 4, 5, are the stresses at G; 3, 4, 6, 7, are the forces at K, of which 6 was the only unknown force. Since each polygon of forces will give us two new forces, we may allow the stress-diagram to

give us W at K by drawing the stress marked 6 parallel to DK. The error in the magnitude of W so obtained provides us with a check on our working. Now we proceed to the hinge F. In this case, because the loads and the frame are symmetrical, the stress in BF can be inferred from the stress in EH, to which it is obviously equal. That leaves only the two stresses marked 10 and 11 undetermined, and these we find from the polygon for F.

If the loads had not been symmetrical we should have calculated the stress in BF by the method used for the stress in EH; that is, by taking moments about the intersection of AD and EC for the equilibrium of the triangle ABC.

There is no need to proceed further. The rest is all straightforward. In the present case, on account of the symmetry of the loads, the stresses in the other half of the frame can be inferred from those already found.

118. Stresses due to Forces applied at Points of the Rods other than the Ends.—When forces are applied to one rod of a framework at points other than the ends, we can find the stresses in the other rods if we replace these forces by two equivalent forces at its hinges and use the method of Art. 110. But this method will not usually give the stresses in the particular rod to which the forces are applied, for it is easy to show that if the applied forces have components along the rod the tension or thrust in the rod is different on opposite sides of a point where a force is applied. Hence the stress is different at different points of the rod, and the method of Art. 110 cannot indicate this since it will only give one stress for each rod.

To illustrate how the stress alters on passing a point where a force is applied let us suppose a weight W is attached to a point C of a vertical rod AB. The tension just below C is equal to the sum of all the downward forces on the portion of the rod below C. The tension just above C is equal to these same downward forces plus the weight W. Thus on passing C there is a sudden alteration in the tension by an amount W. Moreover, if the weight of the rod itself be taken into account, there is a continuous increase in the tension in going up the rod, because the weight of rod below any point increases with the distance from the lowest point.

Nevertheless, although the stress-diagram will not immediately give the stresses in the framework, it can be used to get the actions of each rod on the hinges as we will now show by an example. From the reactions of the hinges on the rods and the known applied forces the stresses in the rods can be calculated.

119. Example of Framework with Forces applied to the Rods.—Suppose we want the actions at the hinges in a triangular framework formed of three equal uniform rods when the only actions on them are their own weights and a supporting force at one corner, A.

Let W denote the weight of one rod.

Now it is clear that the actions of any one rod on its hinges will not be altered if that rod is replaced by a light rod with two equal weights, each equal to $\frac{1}{2}$W attached to its ends. If we make this same substitution for each rod we shall get a different system of stresses in the rods,

but we shall leave the actions at the hinges unaltered. After these substitutions we may consider each hinge and the weights concentrated near that hinge as one particle, and then use the same methods to deal with these compound particles as we have previously used for hinges. Thus in Fig. 48A the particles at B and C are each acted on by W, made up of $\frac{1}{2}$W from each of the rods meeting at those corners. At A a force 2W acts upwards, this being the resultant of half the weights of the rods meeting at A and 3W upwards.

Now $\alpha\beta\gamma$ is the triangle of forces for the equilibrium of the particle at C; $\gamma\beta\delta$ for the particle at A; and $\delta\alpha\gamma$ for the particle at B.

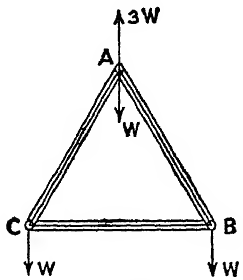


FIG. 48A.

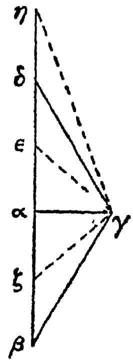


FIG. 48B.

Now the light rod AB, which has taken the place of the given one acts on the particle at B with a force represented by $\overrightarrow{\gamma\delta}$, and this action is transmitted by the end of the rod to the hinge at B. But, in addition to this, the weight $\frac{1}{2}$W attached to the end of AB must be supported by the hinge. Hence the whole action of the rod AB on the hinge at B is the resultant of $\overrightarrow{\gamma\delta}$ and $\frac{1}{2}$W downwards; the whole action is $\overrightarrow{\gamma\epsilon}$, where ϵ is the mid-point of $\alpha\delta$. Similarly the whole action of AC on the hinge at C is $\overrightarrow{\zeta\gamma}$. Again, the action of AB on the hinge at A is the resultant of $\overrightarrow{\delta\gamma}$ and $\frac{1}{2}$W downwards, that is, $\overrightarrow{\eta\gamma}$ where $\overrightarrow{\eta\delta}$ represents $\frac{1}{2}$W. The actions of the two hinges at A and B on the rod AB are $\overrightarrow{\gamma\eta}$ and $\overrightarrow{\epsilon\gamma}$ respectively. The resultant of these is $\overrightarrow{\epsilon\eta}$, which is an upward force balancing the weight of the rod.

120. When forces are applied to the rods of a framework, the method to be used to find the actions of the rods on the hinges may be briefly stated as follows:—

1st Step. Replace the forces on each rod by two equivalent forces at its hinges.

2nd Step. Find the actions of the rods on the hinges due to this new system of forces by the method of Art. 110.

3rd Step. To the action of each rod on a hinge as given by the second step add the force which has been transferred from the rod to that hinge, and this gives the true action of the rod on the hinge.

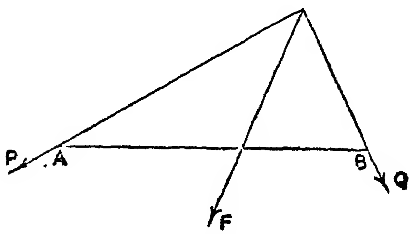


FIG. 49.

It may seem that the above process does not give a definite result because the first step can be performed in an infinite number of ways. Suppose F is a force which has to be replaced by forces at A and B, and let P and Q be one pair of equivalent forces. We could replace F by a pair of forces P_1, Q_1 along any lines through A and B which intersect on the line of action of F. But it is easy to prove that

$$P_1 = P + (\text{a force X along BA})$$
$$\text{and } Q_1 = Q + (\text{a force X along AB})$$

Now, the only effect which the two forces X at A and B along BA and AB respectively would produce on the framework would be a tension in the rod AB. Now suppose the stress-diagram gives a tension T in the rod AB. Then the true action of AB on the hinge at A, due to the forces P and Q, is the resultant of P and a force T along AB. The true action on the hinge in the second case would be the resultant of P_1 and a force $T + X$ along AB. But since P_1 is equivalent to P and $-X$ along AB, the action on the hinge given by both pairs of forces is the same. It follows, therefore, that we may replace F by *any* pair of forces through A and B which are equivalent to F; so that the indefiniteness in the first step does not lead to an indefinite result, but merely gives a greater freedom in the intermediate steps.

EXAMPLES ON CHAPTER IV

1. Draw a triangle ABC with BC vertical, and suppose it to represent a frame of weightless bars. A weight W is hung from A, and BC is kept vertical by being fastened to a wall at two given points X and Y. Find the stresses in AB, AC, and the forces at X and Y caused by W.

[If B is above C there is a tension in AB equal to $\frac{AB}{BC} \cdot W$, and a thrust in AC equal to $\frac{AC}{BC} \cdot W$. The forces at X and Y are separately indeterminate, but the sum of their upward components is W, and their horizontal components form a couple which balances the moment of the weight about X or Y.]

2. ABC is a given triangle in which AB and AC are equal. Let AC and CB be two rods joined by a smooth hinge at C; AB is a string joining the ends A and B. The system stands upright with AB horizontal and C above AB. If a given weight is suspended from C, find the stresses in AB, BC, and CA, putting the weights of the rods out of the question.

Verify your results by considering the case in which ABC is an equilateral triangle.

[If θ denotes one of the equal angles in the triangle, the tension in the string is $-W \cot\theta \cos 2\theta$, and the thrusts in AC, BC are $W \cot\theta$ and $-W\dfrac{\cos 2\theta}{\sin\theta}$ respectively.]

3. A triangular frame of jointed rods ABC, right angled at A, can turn freely about A in a vertical plane. The side AB is horizontal, and the corner C rests against a smooth vertical stop below A. Find (graphically or otherwise) the stresses in the various bars due to a weight W suspended from B.

Determine the stresses numerically when AB = 3 ft., AC = 1 ft., W = 50 lbs. *London Inter. Sci.*

[Tensions of 150 lbs. and 50 lbs. in AB and AC, thrust of $50\sqrt{10}$ lbs. in BC.]

4. A light framework of freely jointed rods, in the form of a right-angled isosceles triangle, is suspended from the right angle. Weights w and $2w$ are suspended from the other two joints. Determine the stresses in the rods. *London B.Sc.*

[If ABC is the triangle, and the weights $2w$ and w are at B and C, then there are tensions $\dfrac{6}{\sqrt{5}}w$ and $\dfrac{3}{\sqrt{5}}w$ along AB and AC, and a thrust $\dfrac{2\sqrt{2}}{\sqrt{5}}w$ along BC.]

5. A framework ABCD is formed of four equal rods of negligible weight and a fifth stiffening rod BD. The frame is suspended by the hinge A, and a weight 130 lbs. is attached to C. Find graphically the stress in BD if the angle at A is 60°.

[About 75 lbs.]

6. A rigid rhombus OABC is formed with five equal light rods composing the sides and one diagonal AC, which is vertical, the joints being smooth hinges. Another equal and similar rigid rhombus OPQR is attached to the first one by the hinge at O, and the two are symmetrical about the vertical through O. The highest points, A and P, are joined by another rod. A weight of three tons is suspended at O, and the whole is supported by vertical forces at B and Q. Find the stresses in AC and AP.

[A thrust of 1·5 tons in AC, and a thrust of 5·2 tons in AP.]

7. Points A, B, C, D, are taken on a straight line such that AB = ½BC = CD. On AB, BC, CD, and on the same sides of these, are described equilateral triangles, AEB, BFC, CGD. EF and FG are joined. The complete figure represents a system of freely jointed light rods in a vertical plane, with AD horizontal and lowest. Supports are placed at A and D, and weights of 4 and 6 tons are hung from B and C respectively. Draw a force diagram for the system. Thence determine the stresses in the rods which meet at F, indicating which are tensions and which are thrusts. *London B.Sc.*

[Thrusts of 4·5 and 5·5 tons in EF and FG respectively, and tensions of 3·75 and 2·02 tons in BF and CF.]

8. AB, BC, CD, are equal lengths on a horizontal line. On the upper sides of these lines equilateral triangles ABX, BCY, CDZ, are drawn. Also

XY and YZ are joined, and the whole figure represents a framework of rods joined by smooth hinges. If loads of 7 and 4 tons are suspended from B and C and the frame supported at A and D, find the stresses in BC, BY, CY.

[Tension of 6·35 tons in BC, tension of 1·155 tons in BY, thrust of 1·155 tons in CY.]

9. ABCD is a straight line trisected at B and C, and BCEF is a square. Let AF, AB, BF, BE, FE, BD, ED be rods forming a freely jointed framework, and let this framework be supported at A and D so that AD is horizontal, and BF vertically upwards. Find, by graphical construction, the thrusts or pulls in all the rods due to a weight W placed at E.

London B.Sc.

[In the following table a thrust is reckoned positive :—

Rod . .	AF	AB	BF	BE	FE	BD	ED
Stress . .	$\frac{\sqrt{2}}{3}$W	$-\frac{1}{3}$W	$-\frac{1}{3}$W	$\frac{\sqrt{2}}{3}$W	$\frac{1}{3}$W	$-\frac{2}{3}$W	$\frac{2\sqrt{2}}{3}$W.]

10. ABOC, PQOR are two similar frames in the form of rhombuses connected by a hinge at O, and stiffened by diagonal rods BC and QR. The corners A and P are attached to two fixed pins in a horizontal line. The frames are in one vertical plane and lie entirely above AP. The rods AB, PQ are each inclined to AP at 30°, and the rods AC, PR, at 60°. If a load of 4 tons is suspended from O, what are the consequent stresses in the rods?

[First show, by considering the equilibrium of one rhombus, that the line of action of the force at the support passes through O.

In the diagonals of the rhombuses there are tensions of 0·758 of a ton, and in all the other rods thrusts of 1·464 tons.]

11. AB, BC, CD, DE, B*b*, C*c*, D*d* are seven equal light rods, the first four horizontal and the last three vertical, all the joints being smooth hinges. The whole frame is kept rigid by rods joining A*b*, B*c*, D*c*, E*d*, *bc*, *cd*. Loads of 5, 3, and 7 tons are supported at B, C, and D respectively, and the frame rests on supports at A and E. Find the stresses in the rods B*c*, BC, CD, D*c*.

[Tensions of 9 tons in each of BC, CD ; thrusts of 2·83 and 1·41 tons in B*c* and D*c*.]

12. ABCD, PBCQ are squares on opposite sides of BC, and E is the centre of the latter square. A framework is made of weightless rods AB, AD, DC, BC, DB, BE, CE, freely joined to one another. A is freely pivoted to a smooth vertical wall, and D presses against the wall below A. A weight W is hung from E. Calculate the stresses in all the rods.

London B.Sc.

[Tensions of $\frac{3}{2}$W, $\frac{1}{2}$W, W, $\frac{1}{\sqrt{2}}$W, in AB, BC, AD, BE; thrusts of $\frac{1}{2}$W, $\sqrt{2}$W, $\frac{1}{\sqrt{2}}$W, in DC, BD, CE.]

13. Five uniform rods form the sides of the rhombus ABCD and the diagonal BD, which has half the length and half the weight of a rod forming a side. If the weight of BD is w, and if the frame is suspended by the hinge at A, find the action on the hinge at B of each of the rods joined to B.

[BD exerts a thrust 1·16w on B and a downward force $\frac{1}{4}w$; AB exerts a tension 3·62w and a downward force w; BC exerts a tension 1·03w and a downward force w.]

14. A and B are two points on a vertical wall to which two rods AD and BC, perpendicular to the wall, are attached by hinges, A being vertically

above B. The weights of these rods are 2 cwts. and 10 cwts. respectively, and act at their middle points. Light rods join BD and DC. The angles DBC, DCB are 30° and 60° respectively. Find the tensions or thrusts in all the rods, and find the action of AD on the hinge at A.

[Tensions of 13·28 cwts. and 5·77 cwts. in AD and DC respectively; thrusts of 2·89 cwts. and 12 cwts. in BC and BD. The action of AD on A is 13·32 cwts., the resultant of the tension 13·28 cwts. and a downward force 1 cwt.]

15. ABCD is a trapezium formed of bars of negligible weight freely jointed at the vertices, the parallel sides being BC and AD; the joints B and D are also connected by a bar. The figure is placed in a vertical plane with the joints A and D resting on fixed smooth pillars of the same height. From the middle point of BC is suspended a load W. Being given that

$$AB = 13,\ BC = 26,\ CD = 15,\ DA = 40,$$

prove that the stress in BD is a tension equal to

$$\tfrac{87}{140}W.$$

London Inter. Honours Math.

CHAPTER V

WORK AND ENERGY

121. Definition of Work.—If a force F, which is constant in magnitude and direction, acts on a particle during its displacement from A to B, then the product of AB and the component of F along AB is called the *work* done by the force in this displacement. If the direction of F makes an angle θ with $\overrightarrow{AB}$, the work done is

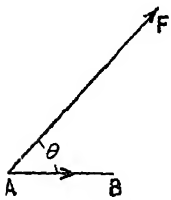


FIG. 50.

$$
\begin{aligned}
W &= F \cos \theta \, AB \\
&= F(AB \cos \theta) \\
&= F \times (\text{the component of } \overrightarrow{AB} \text{ along the line of action of } F)
\end{aligned}
$$

Thus the work done is the product of the magnitude of either of the vectors F or AB and the component of the other parallel to this vector.

If θ is a right angle the work done is zero, and if θ is greater than a right angle the work is negative. The sign to be attached to the work depends, it will be seen, on the *relative* directions of the force and the displacement.

The preceding definition only applies to a constant force. We will now extend the definition to embrace variable forces, when the motion takes place along any curve.

Let the curve be divided into small elements so short that they may each be considered straight. Let ds be the length of one of these elements, and let T be the component force along the tangent to the curve. If dW is the work done in the displacement ds, this work done is

$$dW = T ds$$

Whence

$$\frac{dW}{ds} = T$$

and

$$W = \int T ds$$

EXAMPLE.—*A particle is attracted towards a fixed point with a force which varies inversely as the square of the distance from the point.* It

the particle moves along a straight line AB, *to find the work done by the attraction in the displacement.*

Let P be any point on AB, and let AP $= x$, OP $= r$. Then if the attraction is $\frac{\kappa}{r^2}$

$$T = \frac{\kappa}{r^2} \cdot \cos OPB$$

$$= \frac{\kappa}{r^2} \cdot \frac{c - x}{r}$$

where $c =$ AM and M is the foot of the perpendicular from O on AB.

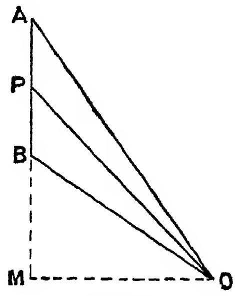


FIG. 51.

Hence, if AB $= l$, OM $= p$

$$W = \int_0^l \frac{\kappa}{r^2} \cdot \frac{c - x}{r} \cdot dx$$

$$= \int_0^l \frac{\kappa(c - x)dx}{\{p^2 + (c - x)^2\}^{\frac{3}{2}}}$$

$$= \left[\frac{\kappa}{\{p^2 + (c - x)^2\}^{\frac{1}{2}}}\right]_0^l$$

$$= \left[\frac{\kappa}{\{p^2 + (c - l)^2\}^{\frac{1}{2}}} - \frac{\kappa}{(p^2 + c^2)^{\frac{1}{2}}}\right]$$

$$= \frac{\kappa}{OB} - \frac{\kappa}{OA}$$

122. Work done by a Couple in a Small Displacement.—Let equal and opposite forces F and $-$F forming a couple, act at points A and B fixed in a rigid body. Suppose the body moves so that AB is displaced to A′B′; then, of course, A′B′ = AB, since the body is rigid.

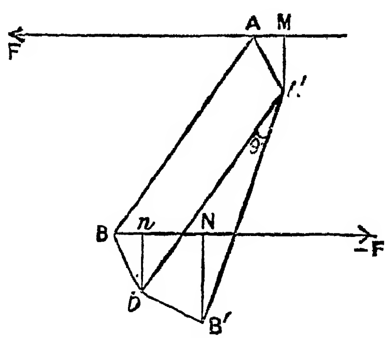


FIG. 52.

A′b is drawn equal and parallel to AB. Also A′M, bn, B′N are drawn perpendicular to the forces of the couple. Then the work done by the forces of the couple is

$$\begin{aligned} W &= F \cdot BN - F \cdot AM \\ &= F(Bn + nN - AM) \\ &= F \cdot nN \end{aligned}$$

since Bn = AM from the geometry of the figure. Thus

$$
\begin{aligned}
W &= \text{Work done by } -F \text{ in the displacement } \overrightarrow{bB'} \\
&= bB' \times (\text{component of } -F \text{ along } \overrightarrow{bB'}) \\
&= A'B' \,.\, \theta\,(\text{component of } -F \text{ along } \overrightarrow{bB'}) \\
&\qquad\qquad \text{approximately, since } \theta \text{ is small,} \\
&= \theta \times (\text{moment of } -F \text{ about } A) \\
&= \theta \times (\text{moment of couple})
\end{aligned}
$$

Thus the work done by a couple in a small displacement of the rigid body on which it acts is simply the product of the moment of the couple and the *angular* displacement of the body in the direction in which the couple turns. In a motion of translation without rotation, such as from AB to A$'b$, no work is done by the couple.

123. Finite Displacement.—The work done by a couple of moment N, whether constant or variable, in any angular displacement from θ_1 to θ_2 of the rigid body to which it is applied, is

$$\int_{\theta_1}^{\theta_2} N d\theta$$

since $Nd\theta$ is the work done in a small displacement. If N is constant this gives $N(\theta_2 - \theta_1)$.

124. Several Couples.—If several couples whose moments are N_1, N_2, N_3, etc., act on a rigid body, the sum of the work done by all the couples is

$$
\begin{aligned}
&\int_{\theta_1}^{\theta_2} N_1 d\theta + \int_{\theta_1}^{\theta_2} N_2 d\theta + \int_{\theta_1}^{\theta_2} N_3 d\theta + \dots \\
&= \int_{\theta_1}^{\theta_2} (N_1 + N_2 + N_3 + \dots) d\theta \\
&= \int_{\theta_1}^{\theta_2} \Sigma N \,.\, d\theta
\end{aligned}
$$

where ΣN is the sum of the moments of the couples, which is equal to the moment of the resultant couple.

125. It has been proved in Art. 71 that a force F acting at any point in a rigid body is statically equivalent to an equal force acting at any other point in the body together with a couple. We will now show that in any displacement of the body, the work done by the force is equal to the work done by the equivalent force and couple.

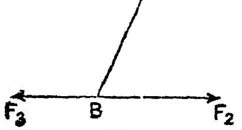


FIG. 53.

Let A be the point of application of a force F_1, B any other point fixed in the rigid body. A force F_2, equal and parallel to F_1, and an equal but opposite force F_3, are introduced at B.

In any displacement whatever the whole work done by the force F_2 and the couple formed by F_1 and F_3 is merely the work done by the forces F_1, F_2, and F_3. But the work done by F_2 and F_3 is zero in any

displacement, because their point of application has the same displacement, and one force is the negative of the other. Thus the work done by the force F_2 and the couple is reduced to the work done by F_1, and this is what we set out to prove.

126. The Sum of the Works done by Several Forces acting at the same Point is equal to the Work done by their Resultant.—Suppose the whole displacement of the point is divided up into a succession of small displacements each of which may be considered straight. Let $\overrightarrow{AB}$ (Fig. 50) be one of these small displacements.

Then the work done by the several forces is equal to the product of OA and the sum of the components of the forces along $\overrightarrow{OA}$. But this sum is equal to the component of the resultant of the forces along $\overrightarrow{OA}$. Hence the sum of the works done by the forces is equal to the product of OA and the component of the resultant along $\overrightarrow{OA}$, which is the work done by the resultant. Thus the theorem is proved.

127. Work done by any System of Coplanar Forces.—It has been shown in Art. 72 that any system of coplanar forces acting on a rigid body can be reduced to a force through any chosen point in the body together with a couple. It will now be shown that the work done by the system of forces is equal to the work done by the force and couple to which the system reduces.

Each given force can be reduced to a force through the chosen point and a couple. Now the work done by all the forces is equal to the work done by all the couples together with the work done by all the forces acting at the chosen point. But the work done by all the couples is equal to the work done by the resultant couple, and the work done by all the forces acting at the chosen point is equal to the work done by the resultant of all these forces by the last article. Thus the theorem is proved.

128. Analytical Expression for the Work.—We will now express in analytical form the work done by a system of coplanar forces F_1, F_2 etc. Let axes OX, OY, fixed in space, be taken, and let any force F_1 acting on the rigid body be replaced by a couple N_1 and two component forces X_1 and Y_1 parallel to the axes and acting at a point fixed in the rigid body whose co-ordinates are x and y. Suppose any line in the body, parallel to the plane of the axes, turns through an angle $d\theta$ while x and y increase by dx and dy. Then the whole work done in any displacement is

$$\int X dx + \int Y dy + \int N d\theta$$

where

$$X = X_1 + X_2 + \text{etc.}$$
$$Y = Y_1 + Y_2 + \text{etc.}$$
$$N = N_1 + N_2 + \text{etc.}$$

129. In finding the work done by a force, any point in the line of action of the force may be regarded as its point of application. For if A and B are any two points in its line of action, the motion of these

points along the line AB is the same in any displacement of the rigid body in which they are fixed. This follows from the fact that the length AB is invariable—since they are points fixed in a rigid body—and therefore there can be no relative motion of the points A and B along the line AB. Thus, whether we consider the force to act at A or B, the work done will be the same, because the displacement in the direction of the force is the same at both points in any interval of time.

130. If the point of application of a force acting on a rigid body is not fixed in the body, but shifts from point to point, the work done in any small displacement is obtained by multiplying the force by the common displacement, along the line of action of the force, of the particles of the rigid body lying in that line. To make this point quite clear, suppose a force F is applied to a particle A of a rigid body at any instant, and suppose that after a short interval of time A has moved to A′ and the point of application of the force is now B, another particle in the body. $\overrightarrow{Aa}$ and $\overrightarrow{Ab}$ are the components of $\overrightarrow{AA'}$ and $\overrightarrow{AB}$ along the force. Then the work done in this displacement is $F \cdot Aa$, and not $F \cdot Ab$.

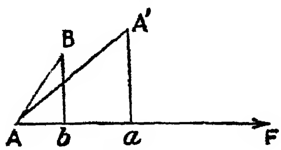


FIG. 54.

131. Rate of doing Work.—Suppose a force F is applied to a particle, which may be part of a rigid body or not, and suppose the displacement of the particle in an infinitely short interval of time dt is ds in a direction making an angle θ with F, then the work done on the particle in this interval by the force F is

$$dW = F \cdot ds \cdot \cos\theta$$

Hence, the rate at which the force does work is

$$\frac{dW}{dt} = F\cos\theta\frac{ds}{dt}$$

$$= FV\cos\theta$$

where V is the velocity of the particle, and $V\cos\theta$ is, of course, the component of this velocity in the direction of F.

If the force F and the displacement ds be each resolved into two components parallel to a pair of perpendicular axes in their plane, the components of the force being X and Y, and those of the displacement dx and dy, then the work done is

$$dW = Xdx + Ydy$$

Therefore $$\frac{dW}{dt} = X\frac{dx}{dt} + Y\frac{dy}{dt}$$

$$= Xu + Yv$$

where u and v are the component velocities of the particle.

It is easy to prove that the two expressions for $\frac{dW}{dt}$ are equal. For let F and ds make angles α and β with OX. Then

$$X = F\cos\alpha, \quad Y = F\sin\alpha$$
$$u = V\cos\beta, \quad v = V\sin\beta$$

Therefore
$$\begin{aligned} Xu + Yv &= FV(\cos\alpha\cos\beta + \sin\alpha\sin\beta) \\ &= FV\cos(\alpha - \beta) \\ &= FV\cos\theta \end{aligned}$$

since
$$(\alpha - \beta) = \pm\theta$$

132. If a couple of moment N acts on a rigid body, and if $d\theta$ is the angular displacement of the body in radians in an infinitely short interval of time, the work done by the couple in this time is

$$dW = Nd\theta$$

Hence the rate at which the couple does work is

$$\begin{aligned} \frac{dW}{dt} &= N\frac{d\theta}{dt} \\ &= N\omega \end{aligned}$$

where ω is the angular velocity of the body in radians per unit time.

The rate at which a body does work is called its *power*. A practical unit of power is a *horse-power*, which is taken as 33,000 foot-lbs. per minute, and is written one H.P.

133. Work done by the Weight of a Particle in any Displacement.—Suppose a particle of weight w travels along any curve from a point at a height z_0 above some fixed horizontal plane to a point at a height z_1 above the same plane. To show that the work done by the weight is $-w(z_1 - z_0)$.

Let z be the height of the particle at any instant. Then dz will denote the vertical displacement of the particle corresponding to any small displacement along the curve. The work done in an upward displacement dz is the product of the whole force w and the component $(-dz)$ of the displacement in the direction of the force; that is, the work done is $-wdz$. Hence the work done in the whole displacement is

$$-\int_{z_0}^{z_1} wdz = -w(z_1 - z_0)$$

This is the work done by the weight. The work done by the lifting force, if this force just balances the weight, is the negative of the above, namely, $w(z_1 - z_0)$.

134. Total Work done by the Weights of a Number of Particles during given Vertical Displacements.—Suppose the weights of the particles are w_1, w_2, w_3, etc., and their initial and final heights are z_1, z_2, z_3, etc., and z_1', z_2', z_3', etc. Then the total work done is

$$\begin{aligned} &-\{w_1(z_1' - z_1) + w_2(z_2' - z_2) + w_3(z_3' - z_3) + \ldots\} \\ &= -\{w_1z_1' + w_2z_2' + w_3z_3' + \ldots - (w_1z_1 + w_2z_2 + w_3z_3 + \ldots)\} \end{aligned}$$

But if $\bar{z}$ and $\bar{z}'$ are the initial and final heights of the centre of gravity of the particles,

$$(w_1 + w_2 + w_3 + \ldots)\bar{z} = w_1z_1 + w_2z_2 + w_3z_3 + \ldots$$
$$(w_1 + w_2 + w_3 + \ldots)\bar{z}' = w_1z_1' + w_2z_2' + w_3z_3' + \ldots$$

Hence the work done

$$= -(w_1 + w_2 + w_3 + \ldots)(\bar{z}' - \bar{z})$$
$$= -(\bar{z}' - \bar{z})\Sigma w$$

where Σw is the sum of the weights.

Thus the work done is the same as if the whole weight were concentrated at the centre of gravity and had the same displacement as that point. This result is true whether the particles form part of one body or are absolutely independent of each other.

As an example, suppose a weight W of liquid is poured from one vessel into another at a lower level. The work done by gravity on the liquid during the whole process is equal to the product of W and the vertical drop of the centre of gravity of the liquid.

134*a*. Examples on Work.

Example 1.—*A magnet of length* l *and pole strength* m *is situated in a uniform field of force of strength* H. *To find the work done by the forces in the field as the magnet is rotated through two right angles from the equilibrium position.*

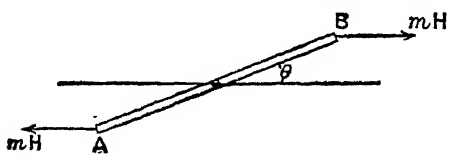


Fig. 55.

There is a force mH acting at each end in opposite directions parallel to the lines of force. Suppose θ is the angle through which the magnet has turned at any instant. Then the two forces form a couple whose moment is $m\mathrm{H}l \sin \theta$. Hence the work done

$$= -\int_0^\pi m\mathrm{H}l \sin \theta \, d\theta = -2m\mathrm{H}l$$

Otherwise.—We might have found the work done by each force and added the results. Thus the work done by the constant force mH acting at one end is equal to mH multiplied by the displacement of that end in the direction of the force, that is $m\mathrm{H}(-l)$. Thus the whole work is $-2m\mathrm{H}l$.

Example 2.—*The flywheel of an engine makes* 100 *revolutions per minute. A brake is applied to the wheel, the moment of the friction about the axis of the wheel being* 10,000 *foot-lbs. Find the horse-power of the engine if the speed remains constant.*

The work done per minute, by Art. 132,

$$= 10{,}000 \times 200\pi \text{ ft.-lbs.}$$

Hence the power of the engine

$$= \frac{10,000 \times 200\pi}{33,000} \text{ H.P.} = 190\tfrac{1}{2} \text{ H.P.}$$

135. Potential Energy.—For many kinds of forces in nature it is found that the work done in any displacement of the bodies on which they act depends only on the initial and final positions of the bodies, and not at all on the path taken between these positions. That is, if we take some fixed position as starting-point, the work done by any of these forces in bringing a body to any other position, whose co-ordinates referred to three axes in space are x, y, and z, depends only on these co-ordinates, and is therefore a function of x, y, and z. If W denotes the work done in bringing the body from the starting-point to any other point, there is a single value of W for every point in space. This is expressed mathematically by saying that W is a single-valued function of the co-ordinates of the point. We shall call W the work function.

It is convenient to introduce another function V, such that V + W is constant for all points.

V is called the *Potential Energy* of the body. Strictly we ought to call it the potential energy of the body and the forces, since there could be no potential energy without the forces.

Since V + W is constant, it follows that, if an amount of work δW is done in any displacement, there is a loss of potential energy of amount δW. If δV denotes the increase of potential energy, we have

$$\delta V = -\delta W$$

Since we shall need only differences of potential energy in different positions, we can choose any position we like as the position of zero potential energy.

136. Criterion for the Existence of a Potential or Work Function.—We shall now show what test can be applied to discover whether a given force or system of forces have a potential or work function.

Let the force or forces acting on any particle be resolved along three mutually perpendicular axes, and let the components of all the forces along these axes be X, Y, and Z. Then the work done in a small displacement whose components are dx, dy, and dz is

$$dW = X dx + Y dy + Z dz$$

Now if a function $f(x, y, z)$ can be found such that

$$^{1}\ \frac{\partial f}{\partial x} = -X, \quad \frac{\partial f}{\partial y} = -Y, \quad \frac{\partial f}{\partial z} = -Z \quad . \quad . \quad . \quad (a)$$

the expression for dW takes the form

$$dW = -\left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz\right)$$
$$= -df$$

where df denotes the whole increase in $f(x, y, z)$ due to the increases dx, dy, dz.

[1] For brevity f is written for $f(x, y, z)$ in several places.

Therefore $$W = -f(x, y, z) + C$$

and the potential energy is

$$V = f(x, y, z)$$

if we drop the useless constant.

It is evident, then, that if the equations (α) are satisfied a potential function does exist. Now the conditions that X, Y, and Z should have the forms given by equations (α) are the following three :—

$$\left.\begin{aligned} \frac{\partial X}{\partial y} &= \frac{\partial Y}{\partial x} \\ \frac{\partial Y}{\partial z} &= \frac{\partial Z}{\partial y} \\ \frac{\partial Z}{\partial x} &= \frac{\partial X}{\partial z} \end{aligned}\right\} \quad \ldots \ldots \quad (\beta)$$

If the forces are all parallel to the xy plane we need only the first of these conditions, since both sides of each of the other two equations are zero because Z is zero everywhere, and X and Y are not functions of z.

We will work a few examples to show how the potential function is found in particular cases.

EXAMPLE 1.—*Find the potential function, if there is one, for the forces whose components are* $X = kx$, $Y = ky$, $Z = kz + c$, *where* k *and* c *are constants.*

Here $\frac{\partial X}{\partial y}$, $\frac{\partial Y}{\partial x}$, etc., are all zero. Hence the conditions (β) are satisfied; that is, a potential function does exist. Therefore, if V denotes this function,

$$\frac{\partial V}{\partial x} = -kx, \qquad \frac{\partial V}{\partial y} = -ky, \qquad \frac{\partial V}{\partial z} = -kz - c.$$

These three equations show that

$$V = -\tfrac{1}{2}k(x^2 + y^2 + z^2) - cz$$

which is the potential function required.

If $k = 0$ and $c = -w$ in this example, we get the potential energy of a body due to its weight. The axis of z is vertical, the positive direction being upwards. Then

$$V = wz$$

EXAMPLE 2.—*Suppose the force is an attraction towards (or repulsion from) a fixed point, and the magnitude of the force is a function of the distance* r *from the fixed point. To show that a potential function exists and to find it.*

Taking the origin at the fixed point, and taking the magnitude of the force, directed away from the origin, to be $F(r)$, we get

$$X = \frac{x}{r}F(r), \qquad Y = \frac{y}{r}F(r), \qquad Z = \frac{z}{r}F(r),$$

because the force makes angles with the co-ordinate axes whose cosines are $\frac{x}{r}$, $\frac{y}{r}$ and $\frac{z}{r}$.

Now
$$\frac{\partial X}{\partial y} = x\frac{\partial}{\partial y}\left\{\frac{1}{r}\cdot F(r)\right\}$$
$$= x\frac{d}{dr}\left\{\frac{1}{r}\cdot F(r)\right\} \times \frac{\partial r}{\partial y}$$

But
$$r^2 = x^2 + y^2 + z^2$$

Differentiating with respect to y, keeping x and z constant, we get

$$2r\frac{\partial r}{\partial y} = 2y$$

whence
$$\frac{\partial r}{\partial y} = \frac{y}{r}$$

Therefore
$$\frac{\partial X}{\partial y} = \frac{xy}{r}\cdot\frac{d}{dr}\cdot\left\{\frac{1}{r}\cdot F(r)\right\}$$

Similarly
$$\frac{\partial Y}{\partial x} = \frac{yx}{r}\cdot\frac{d}{dr}\cdot\left\{\frac{1}{r}\cdot F(r)\right\}$$
$$= \frac{\partial X}{\partial y}$$

Thus the first of the conditions (β) is satisfied. And from symmetry it is clear that the other conditions are satisfied also. Hence a potential function exists, and we have now to find it.

Now we know that

$$\frac{\partial V}{\partial x} = -\frac{x}{r}\cdot F(r) = -\frac{\partial r}{\partial x}\cdot F(r)$$

Similarly,
$$\frac{\partial V}{\partial y} = -\frac{\partial r}{\partial y}F(r), \quad \frac{\partial V}{\partial z} = -\frac{\partial r}{\partial z}F(r)$$

Now suppose
$$f(r) = -\int F(r)dr$$

Then
$$\frac{\partial}{\partial x}\{f(r)\} = \frac{d}{dr}\cdot f(r) \times \frac{\partial r}{\partial x}$$
$$= -F(r)\cdot\frac{\partial r}{\partial x}$$
$$= \frac{\partial V}{\partial x}$$

It is clear also that

$$\frac{\partial}{\partial y}\{f(r)\} = \frac{\partial V}{\partial y}, \quad \frac{\partial}{\partial z}\{f(r)\} = \frac{\partial V}{\partial z}$$

Hence
$$V = f(r)$$
$$= -\int F(r)dr$$

This is the potential function required. The following are particular cases of the above :—

$$F(r) = kr, \qquad V = -\tfrac{1}{2}kr^2.$$

$$F(r) = -\frac{k}{r^2}, \qquad V = -\frac{k}{r}.$$

$$F(r) = -\frac{k}{r^3}, \qquad V = -\frac{1}{2}\cdot\frac{k}{r^2}.$$

137. If the potential energy of a particle in two positions are V_1 and V_2, the work done in the displacement from the first to the second position is $-(V_2-V_1)$. If the particle, after any series of displacements, comes back to the first position, the work done is $-(V_1-V_1)$, that is, nothing. Hence no work is done on a particle by forces which have a potential when the particle describes any closed curve so as to come back to the starting-point. Now we can prove that the converse of this is true, namely, that if no work is done on a particle when it describes any closed curve, then the forces acting on the particle have a potential. For, let AMPNA be any closed curve. Then

work done along ANP + work done along PMA = 0

Also, because AMPMA is a closed curve,

work done along AMP + work done along PMA = 0

These two equations give

work done along ANP = work done along AMP

Thus the work done in going from A to P by any path is the same; that is, the work depends only on the initial and final positions of the particle. Hence the work done in going from any standard position to any other position (x, y, z) is a function of the co-ordinates only; which means that a work function, and therefore a potential function, exists.

When a system of forces have a potential function they are said to be *conservative* forces.

138. It is clear that forces which reverse their directions when the direction of motion is reversed cannot be conservative. For instance, a force which always acts against the motion, such as fluid resistance and friction between solids, is not conservative. In any displacement whatever such forces do negative work, and therefore the work done in a closed path is a negative quantity and not zero. In such motion energy is lost to the system. That is, the energy represented by the negative work is no longer in mechanical form in the system composed of the bodies exerting the forces and the body on which they act. But

this energy is not annihilated; it exists largely as heat in these bodies, but it is not immediately available for mechanical work.

139. Stable and Unstable Equilibrium.—If, when a body is slightly displaced from an equilibrium position, the forces brought into play are such as will move the body back towards the equilibrium position, then the equilibrium is said to be *stable;* if the forces will move the body further from the equilibrium position the equilibrium is *unstable;* and if no forces are brought into play the equilibrium is *neutral.*

A needle standing on its point with its centre of gravity vertically above the point of support is in unstable equilibrium. A cylindrical rod lying with its axis horizontal on a horizontal table is in neutral equilibrium. But a rod with an elliptic section lying on a horizontal table with its shorter axis vertical would be in stable equilibrium.

It is evident that rest in a position of unstable equilibrium is practically impossible for any length of time. Even supposing a needle could be placed with its centre of gravity exactly above its point, we can imagine numerous sources of disturbance which would quickly displace it from this position and consequently destroy the equilibrium altogether. We need only mention air-currents and the minute oscillations produced in the supporting body by the jolting of distant objects on the earth's surface as some of the disturbances which are constantly at work.

140. Conditions of Stability for a Body with One Degree of Freedom.—Suppose a rigid body is so constrained that only one geometrical quantity need be known in order to fix its position. It is then said to have one degree of freedom Let θ be a geometrical quantity which determines its position, and let us assume what is usually true in practice, namely, that there is a work function. That is, we shall assume that the work done on the body in bringing it from any standard position is given by the equation

$$W = f(\theta)$$

Now let T be the resultant force on the body in the position θ, and let ds denote the displacement, in the direction in which T acts, of the particles of the body lying in the line of action of T, when θ increases by $d\theta$. The work done during this displacement is Tds. Hence

$$dW = Tds$$

that is,

$$f'(\theta)d\theta = Tds$$

Therefore

$$T = f'(\theta)\frac{d\theta}{ds}$$

Now for equilibrium it is only necessary that T should be zero. Hence in an equilibrium position

$$f'(\theta)\frac{d\theta}{ds} = 0$$

Now to avoid a difficulty here we will suppose that θ is so chosen that $\frac{d\theta}{ds}$ is never zero. This is always possible, for there are innumerable geometrical quantities each of which could be used to fix the position

of a body with one degree of freedom. It is only necessary that θ should be a geometrical quantity which continually increases (or continually decreases) as the body is moved in the same direction.

With the above assumption concerning θ, the condition for equilibrium is

$$f'(\theta) = 0$$

The roots of this equation are the values of θ in the equilibrium positions. Let θ_0 denote one of these roots so that $f'(\theta_0) = 0$. We have now to test whether the equilibrium at θ_0 is stable or unstable.

Suppose the body receives a slight displacement from the equibrium position given by $d\theta$. Then the force brought into play is $d\mathrm{T}$ say.

Now the condition for stability is that this force should have the opposite sign to ds, for then the force and the displacement are in opposite directions. Thus for stability

$$\frac{d\mathrm{T}}{ds} \text{ must be negative.}$$

that is,
$$f''(\theta_0)\left(\frac{d\theta}{ds}\right)^2 + f'(\theta_0)\frac{d^2\theta}{ds^2} \text{ must be negative.}$$

But $f'(\theta_0) = 0$. Hence for stability

$$f''(\theta_0)\left(\frac{d\theta}{ds}\right)^2 \text{ must be negative,}$$

that is,
$$f''(\theta_0) \text{ must be negative.}$$

The conclusion is, therefore, that if

$$\mathrm{W} = f(\theta)$$

there is stable equilibrium where $\theta = \theta_0$, provided

$$f'(\theta_0) = 0,$$

and
$$f''(\theta_0) \text{ is negative.}$$

These are exactly the conditions that W should be a maximum when $\theta = \theta_0$.

Similarly it may be proved that if W is a minimum the equilibrium is unstable.

If it happens that $f''(\theta_0) = 0$, the force brought into play in a small displacement $d\theta$ will have to be found by Taylor's theorem. It will still be found, however, that the conditions which make W a maximum or minimum will give stable or unstable equilibrium.

If W is neither a maximum nor a minimum the equilibrium is neutral or unstable. If $d\mathrm{T}$ is absolutely zero the equilibrium is neutral; but if $d\mathrm{T}$ has the same sign as ds for displacements in one direction, and the opposite sign for displacements in the other direction, then the equilibrium is really unstable, because, after any small displacement, the body is certain to get on the unstable side after a short time.

If V denotes the potential energy of the body, we have

$$\mathrm{V} = \mathrm{C} - \mathrm{W}$$

Consequently V is a maximum when W is a minimum, and *vice versâ*. Hence V is a minimum in a position of stable equilibrium, and a maximum in a position of unstable equilibrium.

141. Particle on a Smooth Curve as Illustration.—If the body whose stability is being considered is a particle of weight w which is free to move on any smooth curve under no forces except its weight and the normal reaction of the curve, we get

$$W = -wz$$

where z is the height of the particle above any fixed horizontal plane.

Now we must choose some geometrical quantity which increases or decreases continuously from one end of the curve to the other, and suppose W expressed to terms of that quantity. If the curve is such as that shown in Fig. 56, where one value of x, a horizontal co-ordinate,

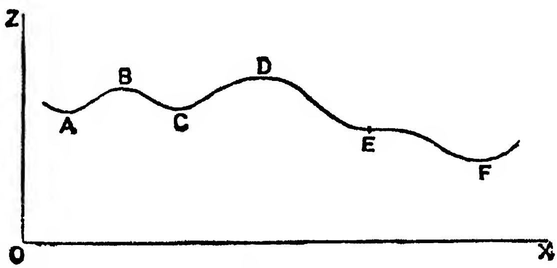


FIG. 56.

corresponds to one and only one point on the curve, we can take x as the geometrical quantity required. Then the condition of equilibrium is

$$\frac{dW}{dx} = 0$$

And for stability

$$\frac{d^2W}{dx^2} = \text{a negative quantity.}$$

These conditions are the same as the following two

$$\frac{dz}{dx} = 0$$

$$\frac{d^2z}{dx^2} = \text{a positive quantity.}$$

Thus the equilibrium is stable at the points A, C, and F, unstable at B and D. At E the first condition is satisfied, but not the second. But it is obvious that if the particle were displaced from E towards D it would fall back and go beyond E towards F, and then leave E altogether. Such a position as E can therefore only be correctly described as unstable.

It should be noted that z could not have taken the place of θ in the general investigation of Art. 140, because $\frac{dz}{ds}$ is zero at several points. The direction of the resultant force and therefore of dT is, of course, along the tangent to the curve at every point.

142. The results in the preceding paragraph can be applied to a rigid body whose centre of gravity is constrained to move along the curve ABF, because the work done by the weight as the centre of gravity is raised to a height z is $-wz$ just as for a particle. Hence a rigid body constrained to move in any way is in stable equilibrium when its centre of gravity is at a minimum height.

We will work out a few examples.

EXAMPLE 1.—*To find whether a solid hemisphere is stable or unstable resting on a horizontal table with its plane end horizontal and uppermost.*

Fig. 57A represents the body in the equilibrium position, and 57B in a disturbed position.

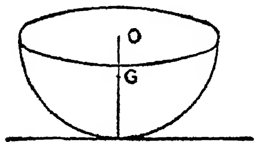


FIG. 57A.

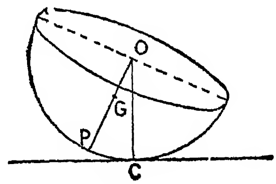


FIG. 57B.

In the second position the height of the centre of gravity G above the supporting plane is

$$OC - OG \cos COP$$

which is greater than

$$OC - OG,$$

the height in the equilibrium position. Hence the equilibrium is stable.

EXAMPLE 2.—*A uniform rod AB rests with its ends on two smooth inclined planes which are at right angles to each other and meet in a horizontal line. If the rod is constrained to remain in a plane perpendicular to the line of intersection of the planes, to find its position of equilibrium, and whether the equilibrium is stable or not.*

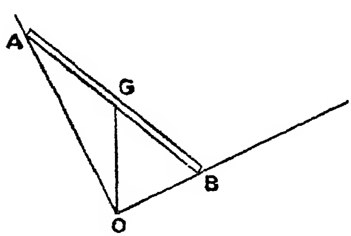


FIG. 58.

As the rod AB moves about subject to the given constraints, it is clear that G describes a circle whose radius is equal to half the length of the rod and with centre at O. The rod will be in equilibrium when G is at the greatest or least height. Thus there is equilibrium when G is vertically above O, but the equilibrium is unstable because

the height is a maximum. There are no other equilibrium positions except when one end of the rod is at O.

EXAMPLE 3.—*Everything is the same as in the last question except that the centre of gravity* G *divides the rod into two parts,* AG *and* GB, *whose lengths are* a *and* b.

Now let GN, the height of the centre of gravity, be denoted by z. Then

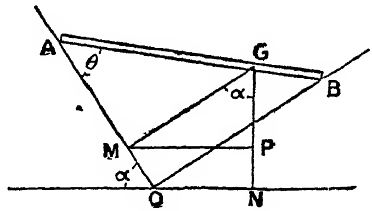


FIG. 59.

$$\begin{aligned} z &= \mathrm{NP} + \mathrm{PG} \\ &= \mathrm{OM} \sin \alpha + \mathrm{MG} \cos \alpha \\ &= \mathrm{BG} \cos \theta \sin \alpha \\ &\quad + \mathrm{GA} \sin \theta \cos \alpha \\ &= b \cos \theta \sin \alpha + a \sin \theta \cos \alpha \end{aligned}$$

Therefore
$$\frac{dz}{d\theta} = -b \sin \theta \sin \alpha + a \cos \theta \cos \alpha$$
$$\frac{d^2z}{d\theta^2} = -b \cos \theta \sin \alpha - a \sin \theta \cos \alpha$$

For equilibrium $\frac{dz}{d\theta} = 0$, that is

$$\tan \theta = \frac{a}{b} \cot \alpha$$

For all possible values of a, b, and α, there is always a value of θ less than a right angle which will satisfy this equation. That is, there is certainly one equilibrium position. But for such a value of θ both $\sin \theta$ and $\cos \theta$ will be positive. It follows, therefore, that $\frac{d^2z}{d\theta^2}$ is negative in the equilibrium position. Hence z is a maximum and the equilibrium unstable.

EXAMPLE 4.—*A uniform rod of length* 2l *has its lower end attached by a light string of length* r *to a point* O, *and it is constrained to pass through a fixed point* A *at a distance* c *vertically above* O. *To show that the rod is stable in the vertical position if*

$$l < \frac{(c+r)^2}{r}$$

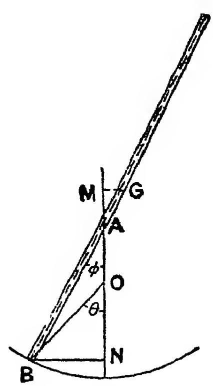


FIG. 60.

Denoting the height of the centre of gravity above O by z, we have

$$\begin{aligned} z &= \mathrm{OM} \\ &= \mathrm{NM} - \mathrm{NO} \\ &= l \cos \phi - r \cos \theta \quad . \quad . \quad . \quad (1) \end{aligned}$$

The potential energy is therefore,

$$V = w(l \cos \phi - r \cos \theta) \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where w is the weight of the rod.

Now
$$\tan \phi = \frac{r \sin \theta}{c + r \cos \theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

so that ϕ may be regarded as a function of θ.

Differentiating (1), we get

$$\frac{dV}{d\theta} = w\left(- l \sin \phi \frac{d\phi}{d\theta} + r \sin \theta\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$\frac{d^2V}{d\theta^2} = w\left\{- l \sin \phi \frac{d^2\phi}{d\theta^2} - l \cos \phi\left(\frac{d\phi}{d\theta}\right)^2 + r \cos \theta\right\} \quad . \quad (5)$$

Again from (3)

$$\sec^2 \phi \frac{d\phi}{d\theta} = r\frac{r + c \cos \theta}{(c + r \cos \theta)^2} \quad . \quad . \quad . \quad . \quad (6)$$

In the vertical position of the rod θ and ϕ are each zero, and therefore

$$\frac{d\phi}{d\theta} = \frac{r}{c + r}$$

Hence in the vertical position

$$\frac{d^2V}{d\theta^2} = w\left\{- l\frac{r^2}{(c + r)^2} + r\right\}$$

For stability this must be positive, because the potential energy must be a minimum That is, for stability

$$l < \frac{(c + r)^2}{r}$$

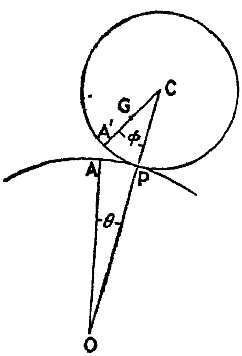


FIG. 61.

143. One body rests on the highest point of a fixed body, the upper body being free to roll without sliding on the lower body. If the possible motion is parallel to a fixed plane, and if the sections of the two bodies by a plane parallel to this fixed plane and passing through their point of contact are circles, to find the conditions of stability.

Referring to Fig. 61, A′ was in contact with A in the equilibrium position; G is the centre of gravity of the supported body. R and r are the radii of the sections of the supporting and supported bodies; and $h = A'G$.

In the disturbed position the height of G above O, the centre of the supporting body, is

$$z = (R + r) \cos \theta - CG \cos (\theta + \phi)$$

Now since the motion has been rolling motion, the arc AP = the arc A′P, that is $R\theta = r\phi$. Hence

$$z = (R + r)\cos\theta - (r - h)\cos\left(1 + \frac{R}{r}\right)\theta$$

The condition for stability is that z should be a minimum when $\theta = 0$, or $\frac{d^2z}{d\theta^2}$ must be positive when $\theta = 0$. Now

$$\frac{d^2z}{d\theta^2} = -(R + r)\cos\theta + (r - h)\left(\frac{R + r}{r}\right)^2 \cos\left(1 + \frac{R}{r}\right)\theta,$$

which becomes, on putting $\theta = 0$,

$$\frac{d^2z}{d\theta^2} = -(R + r) + (r - h)\left(\frac{R + r}{r}\right)^2.$$

Thus, for stability when A′ is in contact with A, we must have

$$(r - h)\frac{R + r}{r^2} - 1 > 0$$

or

$$r^2 + rR - hr - hR - r^2 > 0$$

or

$$\frac{1}{h} > \frac{1}{R} + \frac{1}{r}$$

144. The result of the last article can be applied to bodies which have not circular sections, provided we replace R and r by the radii of curvature of the sections at the points A and A′. For, the stability or instability of the equilibrium depends on whether the path of G is concave upwards or downwards in the neighbourhood of the equilibrium position; that is, it depends on the curvature of the path of G, which clearly only depends on the curvature of the rolling and the supporting curves.

The proof for any curves can be made rigorous, and will be very much the same as that for circles, but it would be rather tedious. We shall therefore let the above general explanation suffice.

If the curvature of either the supporting or supported body is in the opposite direction to that shown in the figure, the corresponding radius of curvature must be considered negative. Thus, if the upper body is concave downwards, and its radius of curvature is r, the condition for stability is

$$\frac{1}{h} > \frac{1}{R} - \frac{1}{r}$$

If either curve is a straight line, the corresponding radius of curvature is, of course, infinite.

145. We will now apply the formula for stability to a few examples.

EXAMPLE 1.—*To test whether a solid hemisphere is stable when standing on a sphere of the same radius* a *with its plane end uppermost.*

Here $\qquad h = \frac{5}{8}a, \quad r = a, \quad R = a$

Therefore $\qquad \frac{1}{h} = \frac{8}{5}\cdot\frac{1}{a}, \quad \frac{1}{R} + \frac{1}{r} = \frac{2}{a}$

Thus $$\frac{1}{h} < \frac{1}{R} + \frac{1}{r}$$

and the equilibrium is unstable.

EXAMPLE 2.—*To show that the hemisphere in the preceding example would be stable when standing with its plane end uppermost on the top of the solid obtained by revolving the parabola* $y^2 = 4ax$ *about the* x-*axis, which is vertical and downward.*

The highest point of the given parabola is at the origin, and we have to find the radius of curvature at that point.

Differentiating both sides of the equation

$$y^2 = 4ax$$

we get $$2y\frac{dy}{dx} = 4a$$

Differentiating again, after dividing by $2y$,

$$\frac{d^2y}{dx^2} = -\frac{2a}{y^2}\cdot\frac{dy}{dx} = -\frac{4a^2}{y^3}$$

Hence, if R denotes the radius of curvature,

$$\frac{1}{R} = \frac{-\frac{d^2y}{dx^2}}{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}} = \frac{\frac{4a^2}{y^3}}{\left(1+\frac{4a^2}{y^2}\right)^{\frac{3}{2}}} = \frac{4a^2}{(y^2+4a^2)^{\frac{3}{2}}}$$

At the highest point, where $y = 0$,

$$\frac{1}{R} = \frac{1}{2a}$$

Then $$\frac{1}{h} = \frac{8}{5a}, \quad \frac{1}{r} = \frac{1}{a}, \quad \frac{1}{R} = \frac{1}{2a}$$

Therefore $$\frac{1}{R} + \frac{1}{r} = \frac{3}{2a} < \frac{8}{5a}$$

and the equilibrium is stable.

146. A body which has more than one degree of freedom may be stable for some motions and unstable for others. For example, a long cylinder resting in a symmetrical position on the top of a sphere would be stable for rolling in the vertical plane containing its axis, and unstable for rolling in the vertical plane perpendicular to this last plane.

EXAMPLES ON CHAPTER V

1. A uniform flexible rope is wrapped round a cylinder whose axis is horizontal, and the length of the rope is equal to the circumference of the cylinder. Its free end is at the end of a horizontal diameter. The cylinder

CHAPTER VI

VIRTUAL WORK

147. Principle of Virtual Work for the Forces acting on a Particle. —If a number of forces act on a particle, then in any conceivable small displacement of the particle the work done by the forces would be equal to the work done by their resultant. If the given forces are in equilibrium, this work in any conceivable displacement would be zero. This is true whether the displacement considered is an actual displacement of the particle or not. It simplifies many problems in mechanics to make use of the work which would be done on a body in an imaginary small displacement, and the work of any force for such a displacement is called the *Virtual Work* of the force. It is assumed that the forces remain constant in magnitude and direction during the displacement. The *principle of virtual work* for a particle may be stated thus: *The virtual work of a system of forces in equilibrium, which act on a particle, is zero for all displacements.*

148. The converse of the principle of virtual work is also true, namely, that *if the virtual work of the forces acting on a particle is zero for all displacements, then the forces are in equilibrium.* For we know that the resultant of such a system of forces can only be a force acting on the particle. And if this resultant were not zero the work done could not be zero for displacements in the direction of the resultant. But by hypothesis the work is zero for all displacements. Hence the resultant must be zero.

149. Principle of Virtual Work applied to Rigid Bodies.—A rigid body, or several rigid bodies, may be regarded as composed of a large number of particles. If, then, several rigid bodies are in equilibrium under the action of any forces, the whole virtual work of the forces acting on all the bodies is zero in all conceivable displacements. For the virtual work for each particle is zero, and the work done by the mutual actions of the particles of the same rigid body is zero, and may, therefore, be ignored in forming the equations of virtual work. To make it clear that the work of the mutual actions of the particles of one body is zero, suppose the particle A exerts a force T on the particle B; then B exerts T in the opposite direction on A. The line of action of these two forces is AB, and since the length AB remains constant (A and B being particles of a rigid body) the displacements of A and B in the direction of AB is the same in any motion. If this common displacement is ds in the direction of T, the work done by A on B is T . ds,

and the work done by B on A is $-T \cdot ds$. Thus the total work done by the action and reaction is zero. Consequently, in forming the equations of virtual work we need only take account of the external forces and the actions between the different rigid bodies. In many cases, too, it is unnecessary to take account of the reactions between the different rigid bodies, as we shall now show.

150. Forces which contribute nothing to Virtual Work.

(*a*) If, in any displacement, two or more bodies remain fixed relative to each other, their mutual reactions do no work, for they are merely equivalent to one rigid body.

(*b*) If the force between two bodies is a mutual pressure at a point of contact, no work is done in any displacement by the action and reaction if the same points of the bodies remain in contact during the displacement. For clearly the work done by the action balances that done by the reaction.

(*c*) If a body slides over a smooth surface the work done by the reaction of the surface is zero, because the displacement is perpendicular to the force. This will be equally true for rigid and non-rigid bodies. It is true, for example, when a flexible string slides over any smooth body round which it is wrapped.

(*d*) If we assume that the only forces between the particles of an inextensible string are those between contiguous particles, it follows that the work done by their mutual actions as the string slides round any smooth surface is zero, because contiguous particles will have the same displacement in the line joining them, which is the line of their mutual actions. Hence an inextensible string may be treated as a rigid body in finding virtual work.

(*e*) The work done by the mutual action between a string and a body to which it is attached is zero for the reason given in (*b*).

(*f*) When a body rolls without sliding on any surface, the work done in a small displacement by the reaction of the surface on the rolling body is zero—or rather, assuming the force constant, the work is a small quantity of the second order in the displacement, and may, therefore, be neglected in comparison with quantities of the first order. To prove this, suppose A′B is a portion of the rolling curve, and AB the curve on which it has rolled. A′ was in contact with A before the displacement. A′O and AC are normals to the two curves at A′ and A. The force acting on the rolling body before displacement, which is the force whose work we are considering, was along CA. The displacement of A′ along CA is approximately AA′, which is nearly the same as A′B × (angle ABA′). Now the angle ABA′ is of the same order of magnitude as $d\theta$, the angle through which the upper body has turned. Also A′B is of the same order as $d\theta$, for its magnitude is

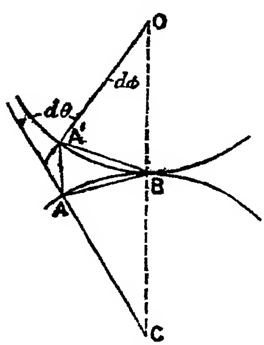


FIG. 62.

approximately $\rho d\phi$, where ρ is the radius of curvature of the rolling body. Hence AA′ is a second order quantity, and therefore the work done by the force along CA is of second order.

(*g*) The mutual actions of two bodies connected by a smooth hinge do no work in any displacement of the hinge, for these actions pass through the centre of the hinge, and this point is fixed relative to both bodies. Hence the points at which the forces are applied in both bodies have the same displacement, and, therefore, the work done by the action is the negative of that done by the reaction.

151. In using the principle of virtual work we keep the magnitude and direction of the forces constant in our assumed displacement, and we suppose the forces to act at points *fixed in the bodies.* Thus the lines of action of the forces will generally move in space during any displacement. Now, if the forces were in equilibrium before the displacement, they will not usually be in equilibrium after the displacement in consequence of this shifting of their lines of action. During part of the displacement, therefore, the forces will be equal to a small couple which will do some work. But this work will be a small quantity of the second order in the displacements. The equation of virtual work will consequently only be correct as far as small quantities of the first order.

Suppose forces P, Q, R, S, etc., act at points of a system of bodies, and suppose dp, dq, dr, ds, etc., are the assumed displacements of these points along the lines of action of the forces; then, if the forces are in equilibrium before the displacement, the equation of virtual work is

$$\mathrm{P}\,.\,dp + \mathrm{Q}\,.\,dq + \mathrm{R}\,.\,dr + \mathrm{S}\,.\,ds + \ldots = 0 \quad . \quad . \quad (1)$$

This equation really means

$$\mathrm{P} + \mathrm{Q}\frac{dq}{dp} + \mathrm{R}\frac{dr}{dp} + \mathrm{S}\frac{ds}{dp} + \ldots = 0 . \quad . \quad . \quad . \quad (2)$$

where $\frac{dq}{dp}$, $\frac{dr}{dp}$, etc., are differential coefficients, which are to be obtained from the geometrical equations connecting p, q, r, etc., which remain true in different positions of the bodies. Equation (2) is accurate, since the second order quantities which should appear in (1) will vanish in (2).

152. We will now work some examples on the principle of virtual work.

Example 1.—*Four equal rods, each of weight* w, *are hinged together to form a rhombus, and the frame is suspended by the hinge at one corner A. A string is attached to A and the opposite corner C to prevent the frame from collapsing. Find the tension in this string.*

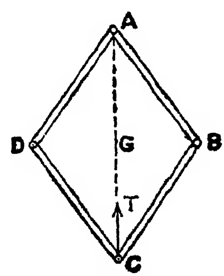


Fig. 63.

In order to use the principle of virtual work in this problem, it is indispensable that the tension of the string should do work in our assumed displacements. We will suppose, therefore, that the string is removed, and that an upward force, T, equal to the tension, is applied

at C. The equilibrium is not affected by this, and we can now allow the force T to do work in a displacement of C.

Let $AC = 2x$. Then $AG = x$, G being the centre of gravity of the four rods. Let x change to $x + dx$. Then the whole work done on the four rods (supposing A to be kept fixed) is

$$4wdx - Td(2x) = 4wdx - 2Tdx$$

the first term being the work done by the weights of the rods as their centre of gravity is lowered by dx, and the second term being the work done by T.

Hence, since the virtual work is zero,

$$T = 2w$$

EXAMPLE 2.—*The same system of rods is suspended in the same way, but is held in shape by a light rod* DB *whose weight we can neglect. To find the thrust in* DB.

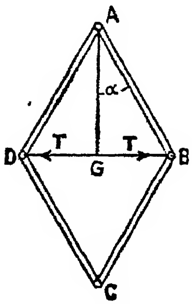


FIG. 64.

Again, suppose the rod DB to be removed and two forces T, equal to the thrust in DB, to be applied outward at B and D. Let $AG = x$, $BG = y$. Now suppose C descends vertically through a small distance, while A remains fixed. We denote the increments in x and y by dx and dy. Then the equation of virtual work is

$$4wdx + 2T \cdot dy = 0 \quad . \quad . \quad . \quad (1)$$

since there is a quantity of work Tdy done at each of the points B and D.

We have now to find a relation between dx and dy from the geometry of the figure. For all such displacements as we have assumed it is clear that

$$x^2 + y^2 = AB^2 = \text{a constant}$$

Hence

$$2xdx + 2ydy = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Equating the two values of $\frac{dx}{dy}$ obtained from (1) and (2), we get

$$-\frac{T}{2w} = -\frac{y}{x}$$

or

$$T = 2w \tan \alpha$$

There is a point in the preceding solution which often puzzles beginners. It might appear from equation (1) that we have assumed the work done by the forces T to be positive. But this is not so. We have expressed the work done by these forces as $2Tdy$, where dy is the *increment* of y corresponding to the increment dx in x. Now equation (2) shows that dy is negative if dx is positive, and consequently the work done by the forces at B and D works out to be a negative quantity. It will thus be seen that, in writing down the work done by the forces, we write each term as if the increments in the geometrical quantities were all positive,

and leave it to the geometrical equations to give the increments their proper signs.

It is worth while to work out the last question in a rather different way. Suppose a changes to $a + da$ in the downward displacement of C. Then

$$x = \mathrm{AD} \cdot \cos a, \quad y = \mathrm{AD} \sin a$$

Therefore $$dx = -\mathrm{AD} \sin a \cdot da, \quad dy = \mathrm{AD} \cos a da$$

The equation of virtual work is therefore

$$-4w \cdot \mathrm{AD} \sin a da + 2\mathrm{T} \cdot \mathrm{AD} \cos a da = 0$$

whence $$\mathrm{T} = 2w \tan a$$

EXAMPLE 3.—*Six rods hinged together form an irregular hexagon symmetrical about a vertical line, the weights being indicated in the figure. The frame is suspended by the hinge at A, and is kept rigid by two equal light rods BF, CE. To find the thrusts in these rods.*

Suppose the thrusts of the light rods FB and EC replaced by forces T_1 at F and B, and T_2 at E and C. To find T_1 we will suppose that θ becomes $\theta + d\theta$, and ϕ remains constant. This allows FB to increase while EC remains of constant length, and thus the forces T_1 will do work while the forces T_2 do no work.

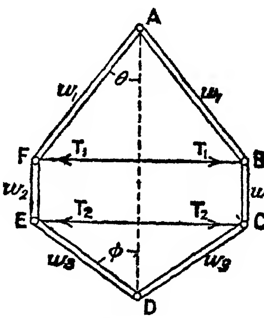


FIG. 65.

Let AB = a, CD = c.

When θ increases by $d\theta$ the rods BC and FE will rise with B and F and will each turn through a small angle of the same order of magnitude as $d\theta$. The vertical displacement of C and E will be the same as that of B and F, together with the displacement due to rotation of the rods BC and FE. But the vertical displacement of C and E due to these rotations will be of the same order of magnitude as $(d\theta)^2$, and may therefore be ignored in the equation of virtual work.[1] It follows, then, that if we take account of the first power of $d\theta$ only, the centre of gravity of the four rods below BF will have the same vertical displacement as B and F. Now the downward displacement of B is

$$+ d(a \cos \theta) = -a \sin \theta d\theta$$

[1] To prove that the vertical displacement due to rotation is of the second order, let the angle through which BC turns be da. Then

$$\mathrm{EF} \sin da + \mathrm{AF} \sin (\theta + d\theta) = \tfrac{1}{2}\mathrm{EC} = \mathrm{AF} \sin \theta$$

whence $$\mathrm{EF} da + \mathrm{AF} \cos \theta \cdot d\theta = 0$$

which shows that da and $d\theta$ are of the same order of magnitude.

Now the vertical displacement of C due to rotation is

$$\mathrm{EF} - \mathrm{EF} \cos da = \mathrm{EF} \cdot \frac{(da)^2}{2}, \text{ neglecting higher powers of } da.$$

This is of the same order as $(d\theta)^2$.

And its horizontal outward displacement is

$$d(a \sin \theta) = a \cos \theta d\theta$$

Thus the centre of gravity of the two upper rods AB, AF, rises through a distance $\frac{1}{2}a \sin \theta d\theta$, and the centre of gravity of the other four rods rises through $a \sin \theta d\theta$. Hence the equation of vertical work is

$$-2w_1(\tfrac{1}{2}a \sin \theta d\theta) - 2(w_2 + w_3)(a \sin \theta d\theta) + 2T_1 a \cos \theta d\theta = 0,$$

from which $$T_1 = (\tfrac{1}{2}w_1 + w_2 + w_3) \tan \theta$$

To find T_2 we suppose ϕ to vary while θ remains constant, the length EC being increased in the process. Then the centre of gravity of the four upper rods receives no displacement of the first order in $d\phi$. The equation of virtual work is now

$$2T_2 d(c \sin \phi) + 2w_3 d(\tfrac{1}{2}c \cos \phi) = 0$$

or $$2T_2 \cos \phi d\phi - w_3 \sin \phi d\phi = 0$$

whence $$T_2 = \tfrac{1}{2}w_3 \tan \phi$$

EXAMPLE 4.—*In Fig. 66, AB is the connecting rod of a steam engine. The end B slides in a smooth groove BO, and the end A is attached by a smooth hinge to a wheel with fixed centre O. If a force P is applied at B in the direction BO, to find the magnitude of the component perpendicular to OA of the force at A.*

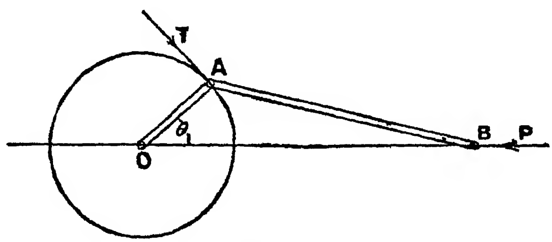


FIG. 66.

Let OA $= r$, AB $= l$, OB $= x$. Let θ increase by $d\theta$, and x by dx. Then the total work done on AB (neglecting its weight) is

$$-P dx - T r d\theta = 0$$

where T denotes the required force at A.

Therefore $$T = -P\frac{1}{r}\cdot\frac{dx}{d\theta}$$

Now $$l^2 = x^2 + r^2 - 2xr \cos \theta$$

Differentiating this with respect to θ,

$$0 = 2x\frac{dx}{d\theta} - 2r \cos \theta \frac{dx}{d\theta} + 2xr \sin \theta$$

Therefore $$\frac{dx}{d\theta} = -\frac{xr\sin\theta}{x - r\cos\theta}$$

Hence $$T = P\frac{x\sin\theta}{x - r\cos\theta}$$

which gives T in any position of the rod.

EXAMPLE 5.—*A heavy chain or string rests partly on a smooth body, with one end attached to the body and the other hanging freely. To find an expression for the tension at any point.*

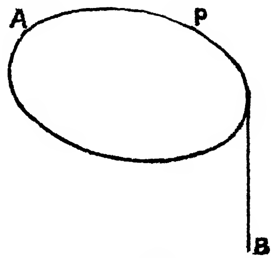


FIG. 67.

Let P be the point at which the tension is required, and let the vertical distance of P above the free end B be denoted by z. Let T be the tension at P, w the weight of unit length of the chain. Suppose the end A is detached from the supporting body, and suppose it is allowed to slide a short distance x over the surface of the body in the direction of the chain at A. The only forces which do work on the portion PB of the chain are, the tension at P and the weight of PB. The work done by the weight of PB in this displacement is clearly just the same as if a portion x of the chain were transferred from P to B. The equation of virtual work for the portion PB is therefore

$$-Tx + wx \cdot z = 0$$

Consequently $$T = wz$$

Thus the tension at any given point is equal to the weight of a portion of the chain whose length is equal to the vertical distance of the point above the free end.

Even if a portion or all of the chain hangs freely without contact with bodies, the tension is given by the expression just obtained. For it would clearly make no difference to the tension in the chain if a smooth curve were introduced exactly fitting against the free portion of the chain. Therefore the tension is wz in the free as well as in the constrained portions of the chain.

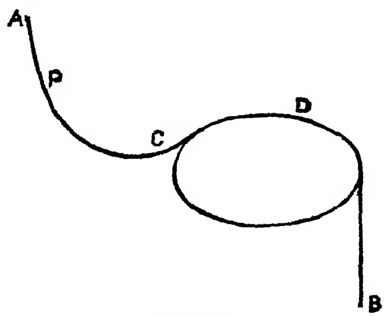


FIG. 68.

Moreover, the tensions would not be affected if any point C or D of the chain in Fig. 68 were to become fixed. Thus, if the part CDB were removed, and only the chain APC were attached to fixed points A and C, the tension at any point P would be proportional to its vertical distance above B, a point

which would not, in this case, be visible in the figure. But althoug we should not know the position of B, we should at least know th the difference of the tensions at any two points of the suspended chai would be proportional to the difference of their heights.

EXAMPLES ON CHAPTER VI

1. AB is a uniform rod (weight W) that can turn freely in a vertical plan round A; a thread BCD is fastened to B, passes over a pulley at C, an carries a weight w at D; W and w are so adjusted that, when AB is at given inclination θ to the vertical, BC is horizontal. If θ receives a smal increase, find the distances through which W rises and w falls. Hence fin the relation between W and w.

[$W = 2w \cot \theta$.]

2. (i) A bar AB, of weight W, is guided by rings at its ends so that A can move on a smooth horizontal rail OX, and B on a smooth vertical rai OY. Employ the principle of virtual work to evaluate the horizontal forc at A necessary to maintain equilibrium when the angle OAB $= \theta$.

(ii) A nut of weight W is mounted on a fixed smooth screw of pitch p, whose axis is inclined at the angle α to the horizontal. What couple is required to keep the nut from moving? *London B.Sc.*

$$\left[\text{(i) } \frac{W}{2}\cot\theta; \quad \text{(ii) } \frac{p\sin\alpha}{2\pi}W.\right]$$

3. A uniform chain hangs between two points A, B, at the same level, and its inclination at these points is α. If the span AB be increased by a small amount k, show that the centre of gravity of the chain is raised through $\frac{1}{4}k \cot \alpha$. [*Manchester University, Honours Engineering B.Sc.*]

4. One shaft is driven by another, inclined to the first, by means of conical cog wheels. If the driving shaft makes n revolutions while the driven one makes m revolutions, find the couple exerted on a driven shaft when a couple of moment M is exerted on the other one, all friction being neglected.

$$\left[\frac{n}{m}M.\right]$$

5. A smooth cone of weight W stands inverted in a circular hole with its axis vertical. A string is wrapped twice round the cone just above the hole and pulled tight. What must be the tension in the string so that it will just raise the cone?

$$\left[\frac{1}{4\pi}W\cot\alpha, \text{ where } \alpha \text{ is the semi-angle of the cone.}\right]$$

6. A smooth sphere of radius r and weight W rests in a horizontal circular hole of radius a. A string is wrapped once round the sphere above the hole and then pulled tight. What tension in the string will just raise the sphere?

$$\left[\frac{W}{2\pi}\cdot\frac{a}{\sqrt{r^2 - a^2}}.\right]$$

7. An endless chain of weight w rests in the form of a circular band round a smooth vertical cone which has its vertex upwards. Find the tension in the chain due to its own weight, assuming the vertical angle of the cone to be 2α.

$$\left[\frac{1}{2\pi}w\cot\alpha.\right]$$

8. The hinges of a gate lie in a line inclined at α to the vertical, and its centre of gravity is at a distance b feet from the line of the hinges. Show that, when the gate has been turned through θ about the line of the hinges from its lowest position, the centre of gravity of the gate is $b \cos \theta \sin \alpha$ feet below its middle position. Thence show that the force which must be applied perpendicular to the gate, at l feet from the line of the hinges, to keep it in equilibrium in this position, is $\frac{b}{l} W \sin \theta \sin \alpha$, where W is the weight of the gate.

9. If every particle of a uniform circular hoop of radius r is repelled from its centre by a force mn^2, where m denotes the mass of the particle and n is a constant, show that the tension in the hoop is $\rho r n^2$, ρ denoting the mass per foot of the hoop.

10. ABCD is a rhombus formed with four rods each of length l and negligible weight, joined by smooth hinges. A weight W is attached to the lowest hinge C, and the frame rests on two smooth pegs in a horizontal line in contact with the rods AB and AD. B and D are in a horizontal line and are joined by a string. If the distance of the pegs apart is $2c$ and the angle at A is 2α, show that the tension in the string is

$$W \tan \alpha \left(\frac{c}{2l} \operatorname{cosec}^3 \alpha - 1\right)$$

11. If the weight of each rod in the last example is w, find the tension in the string when these weights are taken into account.

$$\left[\tan \alpha\left\{\frac{c}{2l}(W + 4w) \operatorname{cosec}^3 \alpha - (W + 2w)\right\}.\right]$$

12. Two equal rods, each of weight wl and length l, are hinged together and placed astride a smooth horizontal cylindrical peg of radius r. Then the lower ends are tied together by a string, and the rods are left at the same inclination ϕ to the horizontal. Find the tension in the string, and if the string is slack, show that ϕ satisfies the equation

$$\tan^3 \phi + \tan \phi = \frac{l}{2r}$$

[The tension is $w(r \sec^2 \phi - \frac{1}{2}l \cot \phi)$.]

13. Two strings, each of length l, are attached to a ceiling, and the lower ends are attached to the ends of a magnet of moment M, length l, and weight W. When the strings are vertical the magnet is in the magnetic meridian, but with its north-seeking pole towards the south. Through what angle will it have to turn before it comes to a position of stable equilibrium? (Assume that the earth's magnetism exerts a couple HM $\sin \theta$ on the magnet when it makes an angle θ with the magnetic meridian.)

[In the position of stable equilibrium either $\theta = 0$ or $\cos \frac{\theta}{2} = \frac{Wl}{4HM}$.]

14. A uniform rod AD, of weight W, has the end A attached to a fixed point, about which it is free to move in any direction. C is the foot of the perpendicular from A on a smooth vertical wall, and B is the point vertically above C such that AB = AD. The end D of the rod is attached by a string to B. When the rod is in equilibrium with the string taut (and therefore AD not in the same plane as ABC), show that the tension in the string is $\frac{BD}{2BC}W$.

15. Three equal spheres are lying in contact on a horizontal plane and are held together by a string which passes round them. A cube of weight

W is placed with one diagonal vertical so that its lower faces touch the spheres, and the cube is supported in this position by the spheres. Show that the tension in the string is $\frac{1}{3}\sqrt{\frac{2}{3}}$W, all friction being neglected.

London B.Sc.

16. The upper ends of four equal light rods, of lengths l, are attached by smooth joints to points on a ceiling at the corners of a square with sides of length b, and the lower ends are attached to points on a board at the corners of an equal square. The board carries a weight W at its centre. The board is then twisted about a vertical axis, so that all the rods are equally inclined to the vertical, and a string is tied round their middle points, which lie on a square with sides x. Find the tension of the string to maintain equilibrium.

$$\left[\frac{Wx}{2\sqrt{l^2 - 2(b^2 - x^2)}}.\right]$$

17. If the four rods of the last example were replaced by n rods attached at the upper and lower ends to regular polygons with sides of lengths b, show that the tension of the string would be

$$\frac{Wx}{n \sin^2 \frac{\pi}{n} \sqrt{\left\{l^2 - (b^2 - x^2) \operatorname{cosec}^2 \frac{\pi}{n}\right\}}}$$

x being the length of a side of the polygon formed by the string.

18. Two uniform discs, each of weight W and radius a, have their planes horizontal. The upper disc is fixed, and the centre of the lower disc is constrained to move on a smooth vertical axis through the centre of the upper disc. The discs are connected by a very large number of equal light rods of length b attached uniformly round their edges. A light elastic band passes round the rods halfway between the discs. Find the tension in this band if there is equilibrium with the rods all inclined at the angle θ to the vertical.

Manchester University.

$$\left[\frac{W}{\pi b}\sqrt{4a^2 + (4a^2 - b^2)\tan^2\theta}\right]$$

19. The 12 edges of a regular octahedron are formed by rods hinged together at the angles. Two pairs of opposite angles are connected by elastic strings whose tensions are t, t'. Show that the pressure along a rod connecting the extremities of these strings is

$$\frac{1}{2\sqrt{2}}(t + t')$$

20. A step ladder has a pair of equal legs which are joined by a hinge at the top, and connected by a cord attached at one-third of the distance from the lower ends to the top. If the weight of the ladder is w acting halfway up, and a man of weight W is two-thirds the way up the ladder, show that the tension in the cord is $\frac{1}{2}(W + \frac{3}{4}w)\tan\alpha$, assuming that the floor is smooth, and that α is the inclination of the legs to the vertical.

21. Four rods of equal weights w form a rhombus ABCD, with smooth hinges at the joints. The frame is suspended by the point A, and a weight W is attached to C. A stiffening rod of negligible weight joins the middle points of AB and AD, keeping these inclined at α to AC. Show that the thrust in this stiffening rod is $(2W + 4w)\tan\alpha$.

22. Four equal light bars are jointed freely so as to form a rhombus ABCD, and the corners A, C, are connected by a light chain. The whole

hangs from A, which is uppermost; and two equal weights W are suspended from B and D. Find (graphically or otherwise) the tension in the chain. *London B.A.*

[The tension is W.]

23. Six equal rods, each of weight w, form a hexagon ABCDEF. AB is fixed in a horizontal position, and the rest of the frame is suspended from AB by means of the joints at A and B and a string joining the middle points of AB and DE, which keeps the other four rods at inclination α to the vertical. Show that the tension in the string is $3w$.

24. ABCD is a framework in the form of a rhombus with AC vertical. A is fixed, and weights w, W, w, are attached at B, C, D, respectively. Besides being acted on by the earth's attraction, each of the weights w is repelled from AC by a horizontal force krw, where $2r =$ BD and k is constant. If there are smooth hinges at the joints, show that, in the equilibrium position, the four rods make an angle with AC whose cosine is $\frac{W + w}{kwl}$, l being the length of a rod.

25. ABCD is a trapezium formed of bars of negligible weight, freely jointed at the ends, the parallel sides being BC and AD; the joints B and D are also connected by a bar. The frame is placed in a vertical plane with the points A and D resting on fixed smooth pillars at the same level. From the middle point of BC is suspended a load W. Being given that AB = 13, BC = 26, CD = 15, DA = 40, prove that the stress in BD is a tension equal to $\frac{37}{240}$W. *London Inter. Sci., Honours Maths.*

26. If the joints A and D, instead of resting on smooth pillars, were fixed by pins driven into a wall, or rested on rough surfaces, would the result of the last question be affected?

[The virtual work method proves that the result would not be affected.]

27. Five equal weightless rods are hinged together at O and rest with their lower ends on a smooth table; and these ends are kept in symmetrical positions at corners of a regular pentagon by an endless string which passes round them and forms a star pentagon. Find the tension in the string when a weight W is suspended from O, having given the inclination, θ, of the rods to the vertical. *Victoria University.*

[$\frac{1}{10}$W tan θ sec 18°.]

28. Three equal smooth spheres, each of weight w and radius a, rest in a symmetrical position at the bottom of a spherical bowl of radius $(b + a)$. Another equal sphere is put on the top of the first three. Show that the mutual pressure between a pair of the lower spheres is

$$\frac{w}{3\sqrt{6}}\left(\frac{16a}{\sqrt{6b^2 - 8a^2}} - 1\right)$$

CHAPTER VII

FRICTION

152. It is well known from experience that the action between two solid bodies in contact is not always—in fact, not usually—merely a force perpendicular to the surface of separation. The simple fact that a body can rest on an inclined plane under no forces but its weight and the reaction of the plane shows that this reaction has a component along the plane which balances the component of the weight down the plane. Another very common fact of observation is that a horizontal force is needed to keep a body moving with uniform velocity on a horizontal plane, and this, combined with Newton's first law of motion, tells us there must be an equal force resisting the motion. That component of the reaction between two bodies which is in the plane touching both surfaces is called *Friction*.

Friction is a force which resists the sliding of one body over another. We will now give the laws of friction.

153. **Laws of Friction.**

Law I.—When two bodies in contact are in relative motion the friction acting on either body at any point of contact is in the direction opposite to the motion of that point relative to the contiguous point of the other body.

Thus, when a body slides on a rough plane without rotation, the friction at every point of contact is in the same direction, namely, opposite to that of the common velocity.

But if a body on a rough plane is rotating about a fixed axis perpendicular to the plane, the friction exerted at any point of the rotating body is perpendicular to the line joining that point to the axis. If the rough plane and the rotating body have, in addition, a common motion through space, leaving the relative motion unaltered, the direction of friction would be unaltered also, because, by the rule given above, the direction of friction depends only on the relative motion.

Law II.—The friction at any point of contact of two given bodies cannot exceed a certain fraction of the normal pressure between the bodies at that point, which fraction depends on the material of the two bodies and the condition of the two surfaces. The maximum friction is called limiting friction and the fraction is called the coefficient of friction for the two bodies.

Law III.—The coefficient of friction is independent of the intensity of the normal pressure.

It follows immediately from the third law that, if there is a given

pressure between two bodies whose points of contact are in one plane, and whose frictions all act in the same direction, the maximum total friction between the bodies does not depend on the area over which the pressure is distributed.

154. Inferences from the Laws.—When two bodies in contact over a plane are acted on by external forces which, were it not for friction, would cause relative motion of the bodies, no motion will ensue if the resultant of the forces is below a certain magnitude, or rather, if the component parallel to the plane of contact is less than a certain fraction of the whole normal pressure. At every point of contact friction is brought into play to resist the motion. The preceding laws state that, if there is a frictional force F and a normal force R, the ratio $\frac{F}{R}$ cannot exceed a certain fraction (denoted by μ) which is constant for two given bodies; but it may have any value from zero up to μ. If the sum of the components of the external forces parallel to the plane of contact is gradually increased while the normal pressure remains constant, an instant will come when the friction at every point, except perhaps one, is the maximum friction, and any further increase in the applied forces will cause relative motion. When motion does finally take place, it may be one of rotation about a fixed axis perpendicular to the plane of contact, and the friction at the point where the axis of rotation meets the plane of contact may not be limiting friction, but there must be limiting friction at all the other points of contact. Just before motion the direction of friction at all points is just the same as when motion begins.

155. When a body is not on the point of motion, that is, when a finite increase may be given to the resultant of the external forces tending to slide the body while the normal pressure remains constant without causing motion, no definite rule concerning the directions or magnitudes of the frictions at different points can be given. All that is known is that the frictional forces will so arrange themselves as to resist motion, provided an arrangement is possible which does not require the friction at any point to be greater than limiting friction. Frictional forces may, and often do, act when the external forces have no tendency to produce sliding. In such a case the frictions balance among themselves and cause a state of strain in the bodies on which they act. As a simple example, suppose an iron rod of considerable length were placed with its ends resting on two supports at the same height. If there is no friction at first, let us suppose the bar is heated. Then one or both of the ends will slide over the supports owing to the expansion, and while the sliding is taking place friction is acting in the direction opposite to the motion. When the temperature becomes stationary the sliding will stop, but it is clear that if the sliding has been continuous (and not in jerks) there will still be friction acting in opposite directions at the ends. The effect of the friction is a slight shortening of the rod, as will be explained in the chapter on elasticity.

156. After considering the last example, it is easy to see that such a thing as a three-legged stool might be placed on a horizontal floor in such a way that frictions act at each point of support, all the frictions

acting towards one point and forming a system of forces in equilibrium. It is only necessary to strain the stool before putting it down in order to produce these frictions. But whatever frictions may act when they are not needed to prevent motion, immediately forces are applied which would produce sliding if the surfaces were smooth, the frictions rearrange themselves to resist the motion, and just before motion takes place they are arranged in the most effective manner possible to resist that motion. That is, the frictions all act in directions opposite to those in which the points of application are about to move.

157. Angle and Cone of Friction.—The resultant of friction F and the normal pressure R exerted by one body on another at any point of contact will make an angle $\tan^{-1}\frac{F}{R}$ with the normal to the surfaces at that point. This angle may have any value from zero to $\tan^{-1}\mu$, depending on the amount of friction brought into action. This resultant force of either body on the other at a point of contact lies somewhere inside a cone whose axis is the normal to the surfaces, and whose semi-vertical angle is $\tan^{-1}\mu$. The cone is called the *cone of friction*, and the angle $\tan^{-1}\mu$ is called the angle of friction.

158. Work done by Friction.—When friction acts at a point fixed in a body there is no need to add anything to the definition of work given in Art. 121. But when the point at which the friction acts is continuously changing in the body itself, as when the body rolls and slides down a plane, we require an extension of the definition of work. This extension may be stated as follows:—

The work done in a short time dt *by a frictional force* F, *acting at points of a body infinitely near a point* P, *is* Fudt, *where* u *is the component velocity of* P *in the direction of* F.

Perhaps this is more precisely stated by means of the rate of doing work. Thus, if F is acting at P at any instant, the rate at which the friction is doing work at that instant is Fu.

159. If F is constant, the work done in any interval of time from t_0 to t_1 is

$$W = F\int_{t_0}^{t_1} u\,dt$$

If, at the same time, the friction acts at a point fixed in space the integral

$$\int_{t_0}^{t_1} u\,dt$$

is the negative of the length of the curve of the moving body, which has passed the fixed point in the interval of time. Denoting this length by s, we get

$$W = -Fs$$

This result can be applied to a shaft turning in fixed bearings. If r is the radius of the shaft, F the friction at any point, the work done while the shaft turns through θ radians is $-F.r\theta$. The total work done on the shaft by the friction at all the points is $-\Sigma Fr\theta = -\theta\Sigma Fr$,

the summation extending to all points at which friction acts. But ΣFr is the moment of the frictions about the axis of the shaft. Hence the work done is $-M\theta$, where M denotes the moment of the frictions.

160. We will now take an illustration in which both bodies move. A body A slides over a rough body B, which is itself moving in the same direction as A. The velocities of the two bodies are u and v.

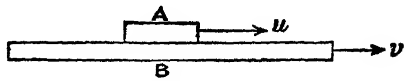


FIG. 69.

If $u > v$ the friction on the upper body is in the opposite direction to u. The work done by this friction is therefore

$$-\int \Sigma F \, . \, udt$$

where ΣF denotes the sum of the frictions at all points of the body. If this sum is constant ($= F_1$, say), the work done by the friction on the upper body is

$$-F_1 \int udt$$

and this result is not altered even if the friction at different points of the body is variable, provided only the sum is constant.

The friction on the lower body acts in the direction of motion, and the work done by it is

$$F_1 \int vdt$$

The total work done on the two bodies by the friction at their surface of contact is thus

$$-F_1 \int (u - v)dt = -F_1 \int Vdt$$

where V is the relative velocity of the bodies, and therefore $\int Vdt$ is the relative displacement.

If v were greater than u, the directions of the frictions would be reversed, and then the total work would be

$$-F_1 \int (v - u)dt$$

which is again the product of $-F_1$ and the relative displacement, taken positive as in the preceding case.

161. In this case, as in more complex cases, the work done by friction of constant magnitude is the negative of the product of the friction and the total relative displacement, this relative displacement being obtained from the infinitely small displacements by *arithmetical*, and not by vector, addition. For example, if the upper body slides over the other body along a straight line of length a and back again to the starting-point (without rotation), the work done by friction on the two bodies is $-2aF$, and not zero. For the purposes of finding work done by friction, the total displacement of a body is the length of the curve it has described, and not the straight line joining the beginning of this curve to the end.

162. We shall now proceed to apply the rules of friction to determine the equilibrium of bodies under frictional forces. We shall assume that the laws stated are quite accurate, although it is proper to warn the student here that they are only fair approximations to the truth. The friction does depend to some small extent upon the intensity of

E*

pressure, and the friction which acts during relative motion is rather less than the maximum friction just before motion takes place. But all these small inaccuracies we shall disregard for two very good reasons: firstly, because the laws are sufficiently accurate for most practical purposes; and secondly, because exact laws are not known.

163. The coefficient of friction for two bodies in contact depends, of course, on both bodies. In no case is it so great as unity. For timber on timber it has values from 0·2 to 0·5, for timber on metals from 0·2 to 0·6, and for metals on metals from 0·15 to 0·25.

164. To show that, if a body is on the point of sliding down an inclined plane under its own weight, the inclination of the plane is equal to the angle of friction.

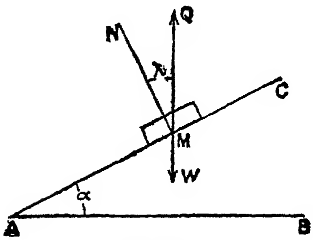


FIG. 70.

Let Q denote the resultant action of the plane on the body, W the weight of the body, λ the angle of friction, NM the normal to the plane. Then since the body is on the point of sliding down, the friction is limiting friction, and Q makes an angle λ with NM. But Q must balance the weight, and must therefore be a vertical force. Then it is obvious from the figure that $\lambda = \alpha$, the inclination of the plane.

165. To find what force must be applied at a given angle θ with the upward direction of an inclined plane in order to drag a body of weight W up the plane.

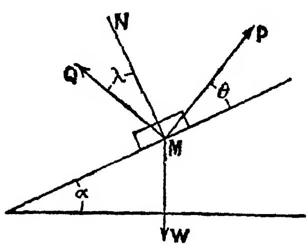


FIG. 71.

Let P be the necessary force, and α the inclination of the plane

Since the body is moving up the plane the friction acts down the plane, and therefore the reaction Q between the plane and the body is on the lower side of the normal MN at an angle λ with it.

The force P is assumed to be just large enough to start motion.

Since a relation between P and W is required, we shall get this relation by resolving perpendicular to the other force Q. Resolving perpendicular to Q, therefore, we get

$$P\cos(\theta - \lambda) - W\sin(\alpha + \lambda) = 0$$

or

$$P = W\frac{\sin(\alpha + \lambda)}{\cos(\theta - \lambda)}$$

This gives the force which will just move the body up the plane when applied at any angle θ. The least force which will move the body up the plane is the least value of P for different values of θ. P will clearly be least when $\cos(\theta - \lambda)$ is greatest, that is, when $\theta = \lambda$, and the value of P for this value of θ is $W\sin(\alpha + \lambda)$.

166. Motion down the Plane.—If the body is just on the point of motion down the plane the friction acts up the plane. The force Q is therefore on the opposite side of the normal from that shown in Fig. 71. It is easy to see, then, that the working of the last article can be applied to the case of motion down the plane under the force P provided we put $-\lambda$ instead of λ. P will now be the force required to prevent motion down the plane, and its value is

$$P = W\frac{\sin(\alpha - \lambda)}{\cos(\theta + \lambda)}$$

If $\alpha < \lambda$ this gives a negative value for P. This means that P is a thrust and not a pull as shown, for in this case the body will need pushing down the plane.

If P has any value between

$$W\frac{\sin(\alpha - \lambda)}{\cos(\theta + \lambda)} \quad \text{and} \quad W\frac{\sin(\alpha + \lambda)}{\cos(\theta - \lambda)}$$

the body will be in equilibrium. But if it is less than the first or greater than the second there will be motion down or up the plane. If the first of these forces is negative a less force will, of course, be a greater negative force, that is, a greater push.

167. A rod or ladder rests in a vertical plane with its ends on a horizontal floor and against a vertical wall respectively. If the rod is just on the point of slipping, to find the position, given that the coefficients of friction are μ at the floor and μ' at the wall, and the centre of gravity divides the rod into two parts whose lengths are in the ratio $m : n$.

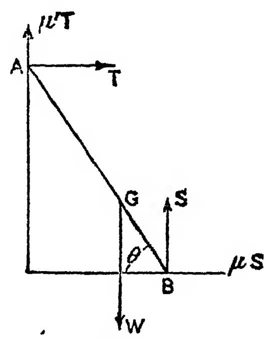


FIG. 72.

This example can be applied to the case of a man at any point of the ladder, the weight being supposed to act at the centre of gravity of the man and the ladder.

In Fig. 72, G is the centre of gravity of the ladder, and AG : GB $= m : n$. Since the ladder is on the point of slipping, there will be maximum friction at both A and B. The forces are shown in the figure.

Resolving vertically and horizontally, we get

$$S + \mu'T = W \quad \ldots\ldots\ldots \quad (1)$$

$$\mu S - T = 0 \quad \ldots\ldots \quad (2)$$

Taking moments about A

$$AG \cdot \cos\theta \cdot W - AB \cdot \cos\theta \cdot S + AB \sin\theta \cdot \mu S = 0 \quad . \quad (3)$$

Now since AG : AB $= m : (m + n)$, equation (3) can be written

$$mW\cos\theta - (m + n)S\cos\theta + (m + n)\mu S\sin\theta = 0 \quad . \quad (4)$$

Substituting in (4) the value of S obtained from (1) and (2), we get

$$mW \cos\theta - (m+n)\frac{W}{1+\mu\mu'}\cos\theta + (m+n)\frac{\mu W}{1+\mu\mu'}\sin\theta = 0$$

which gives $$\tan\theta = \frac{1}{m+n}\left(\frac{n}{\mu} - m\mu'\right) \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Suppose the frictions at A and B are F and P. Then the result obtained can be written

$$\tan\theta = \frac{1}{m+n}\left(n\frac{S}{P} - m\frac{F}{T}\right) \quad . \quad . \quad . \quad . \quad . \quad (6)$$

It is clear that this result is equally true whether the frictions are limiting frictions or not. But if the friction is not limiting friction the fractions $\frac{P}{S}$ and $\frac{F}{T}$ will be less than μ and μ' respectively. But from the form of the right-hand side of equation (6), it is obvious that the smaller either of these fractions is, the greater θ will be. Consequently, the rod will rest in any position nearer the vertical than the one given by equation (5), and the friction will not be limiting friction at both points.

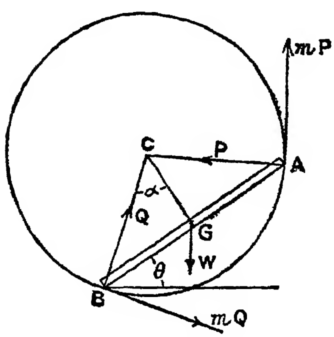


FIG. 73.

168. EXAMPLE.—*A uniform rod rests inside a horizontal cylinder in a vertical plane. To find the positions of equilibrium.*

Let 2α be the angle subtended by the rod at the axis of the cylinder, and θ the angle it makes with the horizontal. Let the normal pressures at the ends A and B be P and Q, and let us denote the frictions at these points by mP and mQ.

By resolving along, and perpendicular to the rod, and taking moments about C, we get

$$(P - Q)\sin\alpha - (mP + mQ)\cos\alpha = -W\sin\theta \quad . \quad (1)$$
$$(P + Q)\cos\alpha + (mP - mQ)\sin\alpha = W\cos\theta \quad . \quad . \quad (2)$$
$$(mP + mQ)r = W\sin\theta \,.\, r\cos\alpha \quad . \quad . \quad . \quad . \quad (3)$$

Multiplying (1) by m^2, (2) by $-m$, and (3) by $\frac{1+m^2}{r}\cos\alpha$, and adding, we get

$$W\{(1+m^2)\cos^2\alpha\sin\theta - m\cos\theta - m^2\sin\theta\} = 0 \quad . \quad (4)$$

which gives $$\cot\theta = \frac{\cos^2\alpha}{m} - m\sin^2\alpha \quad . \quad . \quad . \quad . \quad . \quad (5)$$

When there is limiting friction at both ends m will be equal to μ, the coefficient of friction. The corresponding value of θ is given by

$$\cot\theta = \frac{\cos^2\alpha}{\mu} - \mu\sin^2\alpha \quad . \quad . \quad . \quad . \quad . \quad (6)$$

If m is less than μ the value of θ given by (5) is less than that given by (6). Also it is clear that, by properly choosing the value of m, θ can have any value from zero up to the one given by (6). Hence equilibrium is possible for all inclinations of the rod to the horizontal less than that when limiting friction acts.

We have assumed above that the frictions are mP and mQ, but except when the frictions are limiting frictions it is very unlikely that they will be the same fraction of the normal forces. Actually if the angle θ is given the frictions are not determinate. If we had written mP and nQ for the frictions in equations (1), (2), and (3), we should have had only three equations containing the four unknown quantities P, Q, m, n. From these we could get one relation between m and n by eliminating P and Q, and then it would be possible to give any value to m (less than μ) and calculate n from this relation.

Equation (5) can be written

$$\cot\theta = m\cos^2\alpha\left(\frac{1}{m^2} - \tan^2\alpha\right)$$

from which it follows that the rod will rest in the vertical position if $m = \cot\alpha$. This will be a possible value for m if $\mu > \cot\alpha$. Hence the rod will rest at any inclination to the horizontal if $\mu > \cot\alpha$.

169. Displacement of a Rigid Body parallel to one Plane.

Any displacement of a rigid body in which every particle is displaced parallel to a given plane could be produced by rotation about some axis perpendicular to the plane.

We need only consider the particles of the body in a plane parallel to the plane of motion, for when these are fixed the whole body is fixed.

Let A, B, be the positions of two particles of the body before displacement, A′, B′, their positions after the displacement. The two lines AB and A′B′ will, of course, lie in a plane parallel to the given plane.

Let MO, NO, be drawn perpendicular to AA′, BB′, through their mid-points. Then

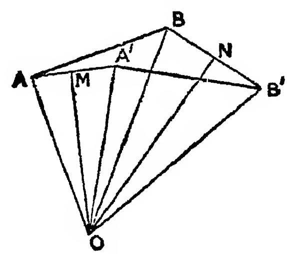


FIG. 74.

$$OA = OA', \quad OB = OB'$$

Also AB = A′B′, because A′B′ is the line AB in a different position. Hence the triangles OAB, OA′B′, are equal in all respects. Consequently the triangle OAB could be brought into the position OA′B′ by rotation about an axis through O perpendicular to the plane of the triangle. Since all the particles have moved parallel to the same plane,

it is clear that rotation about the axis through O would bring the rest of the body into its new position.

There is a case where the preceding argument fails, namely, when AA′ and BB′ are parallel, for then the perpendiculars through M and N never meet. Here it is evident that the displacements of the particles are not merely parallel to a fixed plane, but parallel to a fixed line, and all the displacements are equal as well as parallel.

170. Instantaneous Centre of Rotation.—Since the continuous motion of a body parallel to a fixed plane may be conceived as a succession of infinitely small displacements, and each small displacement can be brought about by rotation about some fixed axis, it follows that we may regard this continuous motion as a series of rotations about a moving axis. The axis of rotation at any instant is called the *instantaneous axis of rotation*, and in dealing with the motion of a plane body in its own plane, the point where the instantaneous axis meets the plane is called the *instantaneous centre of rotation.*

171. The Centrodes.—In the continuous motion of a plane body in its own plane the instantaneous centre will trace out some curve in space and another curve in the body itself. The first of these curves is called the space-centrode, and the second, the body-centrode. It can be shown that the actual motion could be reproduced by the rolling, without sliding, of the body-centrode on the space-centrode. This we will now show.

172. Suppose A, B, C, D, are four points in space about which a body is rotated in succession, and suppose A, *b*, *c*, *d*, are the corresponding points of rotation in the body itself. The body is first turned about A until *b* coincides with B, then about B until *c* coincides with C, and lastly, about C until *d* coincides with D. Now from the geometry of the motion we have the total length of A*b*, *bc*, and *cd*, equal to the total length of AB, BC, and CD. This would be equally true, however many points of rotation we took and however near they may be. But if we take the points infinitely near together we shall get continuous motion, and these centres of rotation will be instantaneous centres of rotation of the moving body. Also the points A, B, C, D, will lie on the space-centrode, and *a*, *b*, *c*, *d*, on the body centrode, and these two curves will touch each other. Thus we arrive at the conclusion that the body-centrode always touches the space-centrode, and the length of any arc of one curve is equal to the length of the corresponding arc of the other, from which it follows that there is no sliding of one curve over the other. The motion is therefore the same as that obtained by rolling the body-centrode on the space-centrode.

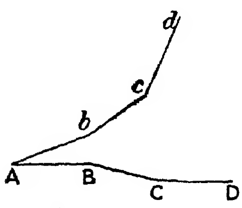


FIG. 75.

When one body does actually roll on another fixed in space it is obvious that the instantaneous centre is at the point of contact.

173. If the directions of motion of two points of a body, moving parallel to a given plane, are known, then the instantaneous centre is

easily found. Let A and B be two points whose directions of motion are AP and BQ. Since A is moving along AP, the centre of rotation must be on the line through A perpendicular to AP. Again, the centre of rotation must be on the line through B perpendicular to BQ. Let AI, BI, be these two lines. Then I is the instantaneous centre.

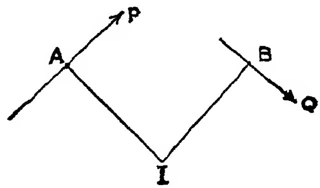


FIG. 76.

Again, we can compare the velocities of all points in the body when the instantaneous centre is known. For if ω is the angular velocity of the body in radians per second, the linear velocity of any point A is IA . ω. Thus the velocities of points are proportional to their distances from the instantaneous centre.

174. A circle rolling inside a circle of twice the radius.

We shall show that any point on the circumference of the rolling circle describes a diameter of the large circle.

The instantaneous centre is I, the point of contact. The centres of the large and small circles are C and O. P is any point on the circumference of the small circle.

P is moving perpendicular to IP, that is, along the line PC. Thus, P always moves towards (or away from) C. If P was originally in contact with P′, it follows that P must describe the straight line P′C, for P cannot be moving towards C and leave this line.

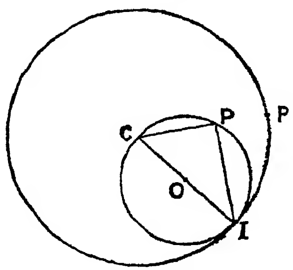


FIG. 77.

Since every point on the circumference of the rolling circle describes a diameter of the fixed circle, we see that in this motion any chord of the rolling circle will move with its ends on two intersecting straight lines.

If the small circle be kept fixed and the large circle made to roll, then every diameter of the large circle will pass through a fixed point on the circumference of the small circle. Thus the diameter through C and P′ will always pass through the point P, which will now be a fixed point in space. Hence if each of two lines of a moving body passes through a fixed point, the motion is equivalent to the rolling of a circle on a circle of half the radius situated inside the rolling circle.

The following is a geometrical proof that C, P, and P′, lie on a straight line when either circle rolls on the other.

The arc IP = the arc IP′, and since the radius of the large circle is twice as large as that of the small circle, it follows that the angle subtended at the centre of the small circle by IP is twice as large as the angle subtended at C by IP′. But the angle ICP is half of the angle IOP; that is, the angle ICP is equal to the angle ICP′, which proves that P′ is on CP produced.

175. A body moves so that two fixed lines in it always touch two circles fixed in space. To find the centrodes.

Let C and D be the centres of the circles; OP, OQ, the lines touching these circles. Through C and D draw CM and DM parallel to PO and QO. Then the two lines CM and DM are fixed in the moving body because their distances from the given lines are constant, being equal to the radii of the given circles.

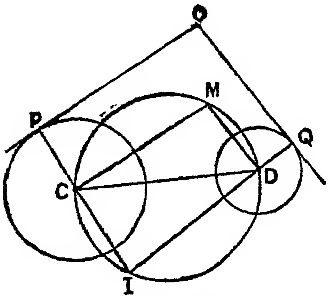


FIG. 78.

The instantaneous centre I is at the intersection of PC and QD.

Now, because the angles at C and D in the quadrilateral CMDI are right angles, a circle will pass through the angular points. And since the angle CMD is equal to the angle POQ, and is therefore constant for all positions of the moving body, it follows that the circle through these points is a fixed circle. It is, in fact, the circle described on the chord CD with the angle POQ in the segment towards O. This circle is the space-centrode.

MI is a diameter of the space-centrode, and is therefore of constant length. But M is fixed in the moving body. Hence I describes a circle in that body with MI as radius. Thus the body-centrode is a circle of twice the radius of the space-centrode.

176. Suppose a body P is kept fixed while a body Q is moved relative to P, and let S and B denote the space-centrode and body-centrode in this motion, the first being a curve described on P, and the second a curve described on Q. Then if the body Q be kept fixed while P is moved, the relative motion being the same as in the previous case, the centrodes will be exactly the same two curves as before; that is, S will be described on P and B on Q. But since P moves and Q is fixed, S will now be the body-centrode and B the space-centrode. This fact will sometimes simplify the question of finding body-centrodes, because it is easier to conceive space-centrodes than body-centrodes. Thus, in the example in the last article, the body-centrode is the space-centrode when two circles attached to a moving plane slide along two lines OP, OQ; that is, it is the space-centrode when a rod CD slides with its ends on two fixed lines MC, MD. Since M is now a fixed point, it is easy to realize in this case that I describes a circle relative to the lines MC, MD.

177. When a body slides on a rough plane it turns, each instant, about an axis perpendicular to the plane which meets the plane in the instantaneous centre of rotation. During this sliding motion the friction at each point of contact acts contrary to the motion, that is, perpendicular to the line joining that point to the instantaneous centre. Since the frictions just before motion takes place are the same as just after the motion begins, it follows that the frictions at the points of

contact of a body which is just about to slide all act along the circumferences of circles having a common centre. This common centre is the point about which the body would begin to rotate if the applied forces were very slightly increased.

EXAMPLE.—*A circular cylinder rests with its axis vertical on a rough horizontal plane, and its weight is uniformly distributed over the base. To find what couple is needed to make it turn about its axis.*

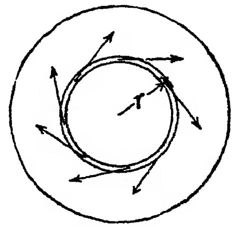


FIG. 79.

Let a denote the radius of the cylinder, W its weight, and μ the coefficient of friction. The moment about the axis of the friction acting on the strip between the circles whose radii are r and $r + dr$ is

$$r \cdot (2\pi r dr)\frac{\mu W}{\pi a^2}$$

since $\frac{\mu W}{\pi a^2}$ is the friction on unit area.

Hence the moment of the whole friction about the axis is

$$\int_0^a \frac{2\pi r^2}{\pi a^2} \cdot \mu W dr = \tfrac{2}{3}a\mu W$$

which is the same as if the whole friction μW acted at a distance $\frac{2}{3}a$ from the centre.

It is clear that the frictions in the above example form a couple, and the result obtained is the moment of that couple.

178. The weight of a body resting on a rough horizontal plane is uniformly distributed over a base of any given shape. If the body turns about an axis perpendicular to the plane of support, to find an expression for the moment of the frictions about the axis.

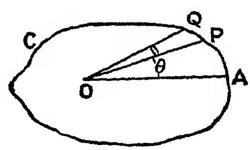


FIG. 80.

Let O be the point where the axis of rotation meets the plane. We will suppose in the first case that O is inside the area of the base. Let APQC be the boundary of the area. Let OP $= r$, angle AOP $= \theta$, and suppose Q is infinitely near P so that OQ $= r + dr$, angle AOQ $= \theta + d\theta$. Let S denote the total area of the base. Then the magnitude of the friction on unit area is $\frac{\mu W}{S}$. When the body turns about O the friction at every point of the small triangle OPQ is nearly perpendicular to OP. Also since the friction on any area is proportional to the area, the resultant friction on the triangle acts at the centre of gravity of its area, namely, at $\frac{2}{3}r$ from O. The magnitude of the friction is $\frac{1}{2}r^2 d\theta . \frac{\mu W}{S}$. Hence the moment of the friction about O is

$$\tfrac{1}{3}r^3 \frac{\mu W}{S} d\theta$$

The moment of all the friction about O is therefore

$$\frac{\mu W}{3S}\int_0^{2\pi} r^3 d\theta$$

Writing $\frac{1}{2}\int_0^{2\pi} r^2 d\theta$ for S this moment becomes

$$\frac{2}{3}\mu W \frac{\int_0^{2\pi} r^3 d\theta}{\int_0^{2\pi} r^2 d\theta}$$

If, instead of being inside the base, O is on the perimeter of the base, the limits of integration are o and π. In this case the moment is

$$\frac{2}{3}\mu W \frac{\int_0^{\pi} r^3 d\theta}{\int_0^{\pi} r^2 d\theta}$$

If O is outside the base the work needs a little modification. In Fig. 81 let AP $= r$, AP$' = r'$. Then the moment of the friction on the strip P$'$PQQ$'$ is the difference of the moments of the friction supposed to act on the triangles OPQ, OP$'$Q$'$. That is, the moment of the friction is

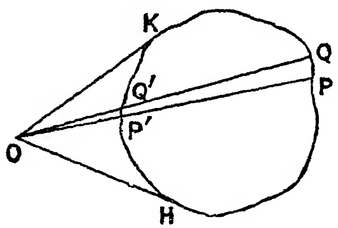


FIG. 81.

$$\frac{1}{3}(r^3 - r'^3)d\theta \frac{\mu W}{S}$$

The moment of the friction on the whole area

$$= \frac{1}{3} \cdot \frac{\mu W}{S} \int (r^3 - r'^3) d\theta$$

the limits for θ being the values of θ at the tangents OH and OK. Also in this case the value of S is

$$\int \tfrac{1}{2}(r^2 - r'^2)d\theta$$

with the same limits for θ.

179. *A uniform rod of length* 2a *and weight* W *rests on a rough horizontal plane, its weight being uniformly distributed along its length. If the rod is just about to move under the action of a force* P *applied perpendicular to the rod at a distance* c *from the centre, to find* P *and the point about which the rod will begin to turn.*

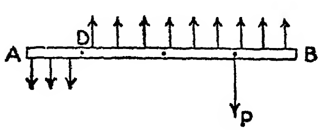


FIG. 82.

The rod will clearly turn about some point in its length. Suppose it begins to turn about a point D at a distance $2x$ from the end B. The frictions on opposite sides of D will act in opposite directions. The frictions acting on BD have a

resultant $\frac{2x}{2a}\mu W$, which acts at the mid-point of BD. Similarly the friction on AD acts through the mid-point of AD and its magnitude is $\frac{2a - 2x}{2a}\mu W$.

Taking moments about the point at which P is applied,

$$\frac{x}{a}\mu W(x + c - a) - \frac{a - x}{a}\mu W(x + c) = 0$$

Hence $$2x^2 - 2(a - c)x - ac = 0$$

The positive root of this equation, which is the one that applies to the problem, is

$$x = \frac{a - c + \sqrt{(a^2 + c^2)}}{2}$$

When $c = a$ this gives $x = \frac{a}{\sqrt{2}}$, and as c decreases from a to 0, x increases from this value to a. But the result in the extreme case where $c = 0$, although correct, is not the complete answer, for the rod could turn about any point in the line of the rod produced.

Resolving perpendicular to the rod to find P, we get

$$P = \mu W\left(\frac{x}{a} - \frac{a - x}{a}\right)$$

$$= \mu W\frac{2x - a}{a} = \mu W\frac{\sqrt{(a^2 + c^2)} - c}{a}$$

180. *A circular hoop is laid on a rough horizontal table, its weight being uniformly distributed. A light string which is wrapped round the hoop is pulled until the hoop just begins to move. To show that it begins to turn about the other end of the diameter through A, the point where the string leaves the hoop.*

It is evident that by reversing the force the friction at every point would be reversed. This could not be the case unless the instantaneous centre were on the diameter through A in both cases. Hence the instantaneous centre must be on this diameter.

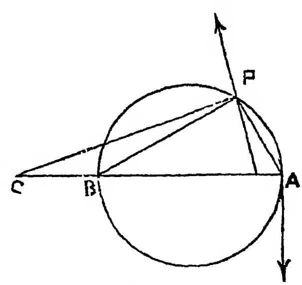


FIG. 83.

Let B be the other end of the diameter through A, and P any point on the circle. Then BPA is a right angle. Now if the instantaneous centre were at C on AB produced, the friction at P would act perpendicular to CP, and its line of action would cut the diameter AB inside the circle. Similarly, the friction at all other points would cut AB inside the circle, and their moments about A would all have the same sign. But for equilibrium

the moments of the friction about A must be zero, and this could not be true if the instantaneous centre were at C. Moreover, it could not be true if the instantaneous centre were inside the circle, for then the frictions would all turn in the opposite direction round A. Hence the instantaneous centre must be at B, and the frictions all act in lines passing through A.

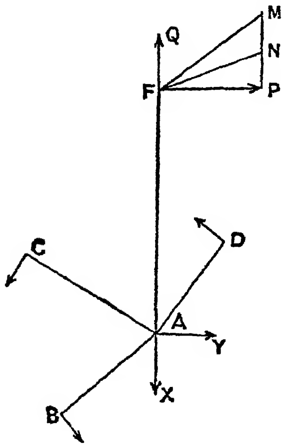


FIG. 84.

181. When a body, which is supported at several points which are in contact with a rough plane, is acted on by external forces, it may begin to turn about one of the points of support. The friction at that point will be less than limiting friction and its direction will be unknown. We will find the conditions for this in the case where the only external force is applied at a given point of support in the body.

Let F be the point at which the external forces are applied, A the point of support about which we are supposing the body to turn. Let P and Q be the components of the external force perpendicular to, and along, AF; and let X and Y denote the components of the friction at A along and perpendicular to FA, as shown in the figure. B, C, D, are the other points of support, and the distances AB, AC, AD, AF, are represented by b, c, d, f. Also Z, R, S, T, are the limiting frictions at A, B, C, D.

Taking moments about A and resolving perpendicular to and along AF, we get

$$fP = bR + cS + dT \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$Y = -P + R\cos\beta + S\cos\gamma + T\cos\delta \quad . \quad . \quad . \quad (2)$$

$$X = Q - R\sin\beta - S\sin\gamma - T\sin\delta \quad . \quad . \quad . \quad . \quad (3)$$

where β, γ, δ, are the angles which AB, AC, AD, make with the direction of X.

Now we know nothing about the friction at A except that it must be less than limiting friction. That is

$$X^2 + Y^2 < Z^2$$

or

$$X^2 < Z^2 - Y^2$$

or

$$(Q - R\sin\beta - S\sin\gamma - T\sin\delta)^2$$

$$< Z^2 - \left\{R\cos\beta + S\cos\gamma + T\cos\delta - \frac{1}{f}(bR + cS + dT)\right\}^2 \quad (4)$$

Since the limiting frictions Z, R, S, T, are all supposed to be known, this last equation gives the limits within which Q must lie. If $Y^2 > Z^2$ it is not possible to satisfy the inequality (4), and therefore the body could not turn about A under the action of a force at F. Let us

suppose, then, that $Z^2 > Y^2$. The inequality (4) now tells us that $(Q - R\sin\beta - S\sin\gamma - T\sin\delta)$ lies between $+\sqrt{Z^2 - Y^2}$ and $-\sqrt{Z^2 - Y^2}$. If Q_1 and Q_2 are the greatest and least values of Q consistent with these condititions, we have

$$Q_1 - R\sin\beta - S\sin\gamma - T\sin\delta = \sqrt{Z^2 - Y^2}$$
$$R\sin\beta + S\sin\gamma + T\sin\delta - Q_2 = \sqrt{Z^2 - Y^2}$$

For turning about A, Q must be less than Q_1 and greater than Q_2. That is

$$\left.\begin{aligned} &Q < \sqrt{Z^2 - Y^2} + R\sin\beta + S\sin\gamma + T\sin\delta \\ \text{and}\quad &> -\sqrt{Z^2 - Y^2} + R\sin\beta + S\sin\gamma + T\sin\delta \end{aligned}\right\} \quad . \quad . \quad (5)$$

These inequalities, combined with equation (1), are the conditions necessary for turning about A under a force at F. There is only one possible value for P, and Q must lie between the limits given by (5).

If FP is drawn to represent P, PN and PM to represent Q_2 and Q_1, the resultant force at F which will cause rotation about A will be represented by a line drawn from F to any point in MN lying between M and N.

182. We will apply the method of the last article to the case of a body on three supports at A, B, and F, the limiting friction at the three points being each equal to R. We shall find what force applied at F will make the body just turn about A, given that FA = FB.

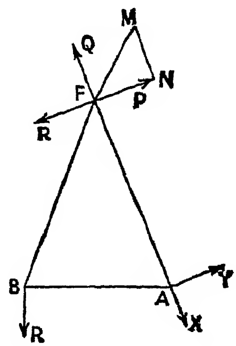


Fig. 85.

Let $FA = f$, $AB = b$, angle $AFB = 2\alpha$.

By moments about A

$$fP = bR + fR$$

whence $$P = R(2\sin\alpha + 1) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Resolving perpendicular to and along AF,

$$Y = -P + R + R\sin\alpha \quad . \quad . \quad . \quad . \quad . \quad (2)$$
$$X = Q - R\cos\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The condition that A should remain fixed is

$$X^2 < R^2 - Y^2$$

or $$(Q - R\cos\alpha)^2 < R^2 - (-R\sin\alpha)^2$$
$$\textit{i.e.} \quad < R^2\cos^2\alpha$$

From this inequality we get

$$\left.\begin{aligned} &Q > 0 \\ \text{and} &< 2R\cos\alpha \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Thus P is given by (1) and the limits of Q are given by (4). If FN represents $R(2 \sin \alpha + 1)$ and NM represents $2R \cos \alpha$, then the resultant applied force at F may be represented by any line joining F to points in NM.

183. If the force applied at F does not lie within the limits found in the last article, it can be shown that it is possible for the body to begin to turn about some point in BA produced or AB produced.

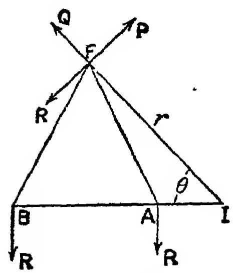


FIG. 86.

It is only necessary to show that, if we suppose the body to turn about some point I in BA produced, the frictions brought into play can be balanced by a force at F.

By taking moments about I and resolving along and perpendicular to IF, we get

$$Pr = r \cos\theta \,.\, 2R + rR \quad . \quad . \quad (1)$$
$$Q = 2R \sin\theta \quad . \quad . \quad . \quad . \quad . \quad (2)$$
$$P = R + 2R\cos\theta \quad . \quad . \quad . \quad (3)$$

The equations (1) and (3) give the same value of P. Hence the three equations are consistent, and the force whose components are P and Q would begin to turn the body about I.

184. **Condition for turning about an unsupported point.**—To find the condition that a body should just turn about a point which is not a point of support, we have only to put $X = 0$ and $Y = 0$ in Art. 181, since the friction at the instantaneous centre will be zero in this case. With these values of X and Y the equations (1), (2), and (3), of that article will give the conditions that the body should turn about A—which is not a point of support—under the action of the force whose components are P and Q.

It will be noticed, however, that we are left with three equations from which to find P and Q, if the positions of both A and F are given. But two quantities cannot generally satisfy three equations. This only means that, by applying a force at a particular point, we cannot make the body turn about any axis we choose, however we adjust the magnitude and direction of the force. But if, in addition to P and Q, a couple whose moment is N is applied to the body, the moment N would appear in equation (1), and then we should have three equations to determine three unknown quantities P, Q, and N. That is, given the point about which the body is to turn we can find the system of forces necessary to produce the motion, and there is only one possible value for P, Q, and N.

But suppose P and Q are given and the instantaneous centre is wanted. The equations (2) and (3) will contain known quantities, and the two unknown co-ordinates of the instantaneous centre referred to any axes we may choose. Hence these two equations are sufficient to determine the position of the instantaneous centre. But now equation (1) has to be satisfied, and it will not usually be satisfied unless we introduce a couple N. When this couple is introduced equation (1)

will determine its moment. Thus the couple N, which, in addition to the given component forces P and Q, will cause the body just to begin to move, cannot be chosen arbitrarily, but is deducible from P and Q.

If, however, only the direction of the force at F were given, it would not be necessary to introduce the couple N into the equations. For then there would be three unknown quantities without it, namely, the resultant force and the two co-ordinates of the instantaneous centre.

Although it is theoretically possible to find the instantaneous centre of initial motion when the body is acted on by a force with given components P and Q, or by a force acting in a given direction at a given point, yet in practice the problem is a difficult one. If there were more than three points of support the equations for finding the instantaneous centre would not be of much use.

185. We will return to the problem of the isosceles triangle considered in Art. 182, and find all the positions of the instantaneous centre when the only external force is applied at F.

Let the frictions at A and B act along AC, CB. By taking moments about F for the equilibrium of the body, we find that the moments of these frictions at A and B are equal in magnitude. That is

$$R \,.\, AF \,.\, \sin FAC = R \,.\, BF \,.\, \sin FBC$$

and since AF and BF are equal, it follows that angle FAC = angle FBC. Consequently the points FCBA lie on a circle.

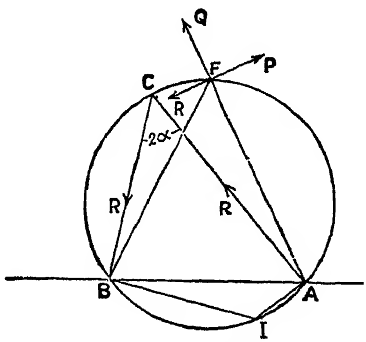


FIG. 87.

Now the instantaneous centre is at I, the intersection of the perpendiculars to AC and BC through A and B respectively. The point I lies, therefore, on the circle through A, B, C, F, and is at the opposite end of the diameter through C.

The instantaneous centre I could not possibly fall on the arc of the circle above AB, for then the frictions at A and B would both turn in the same direction about F, and the sum of their moments would not be zero. But I may fall anywhere on the arc below AB. We will now find P and Q.

Let the equal angles FAC, FBC, be denoted by θ. Then, resolving along and perpendicular to AF, we get

$$\begin{aligned} Q &= R\cos(2\alpha - \theta) - R\cos\theta \\ &= 2R\sin\alpha\sin(\alpha - \theta) \\ P &= R + R\sin(2\alpha - \theta) + R\sin\theta \\ &= R\{1 + 2\sin\alpha\cos(\alpha - \theta)\} \end{aligned}$$

There is yet another way in which the moments about F of the frictions at A and B could be equal. The lines of action of these

frictions might pass on opposite sides of F. But if these lines were to intersect, they would do so on the bisector of the angle at F (produced if necessary), and the corresponding instantaneous centre would be on that line. Now this would make the frictions have equal, instead of opposite, moments about F, and therefore equilibrium would not be possible. But there is still another case to examine. The frictions might not act in intersecting lines, but in parallel lines. To make the moments of the frictions about F equal and opposite in sign, the lines of these frictions would have to be perpendicular to AB, and the instantaneous centre anywhere on AB produced or BA produced, but not between A and B. This agrees with the result of Art. 183.

Thus a force applied at F may make the triangle begin to turn about any point on AB produced, or BA produced, or on that arc of the of circumscribing circle which lies below AB, or about A or B

188. A string or belt wrapped round a rough cylinder of any section.

Suppose a rough string is wrapped round a rough cylinder of any section, and is just about to slip when acted on by no forces but the tensions at its ends.

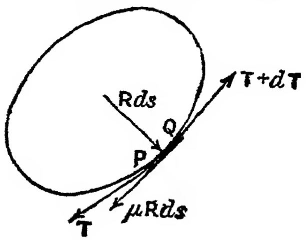


FIG. 88.

Let us consider the equilibrium of an infinitesimal portion PQ whose length we will denote by ds, s being the length of the string from its first point of contact with the cylinder up to the point P. R is the normal pressure per unit length between the string and the cylinder at any point. Hence μR is the friction per unit length when the string is just about to slip. The normal pressure and the friction on ds are therefore Rds and μRds, neglecting all powers of ds above the first. Let $d\psi$ denote the angle between the tangents at P and Q, and $d\phi$ the angle between the normal at P and the force Rds. It is obvious that $d\phi$ is less than $d\psi$.

Resolving along and perpendicular to the tangent at P, neglecting the weight of PQ, we get

$$(\mathrm{T} + d\mathrm{T}) \cos d\psi - \mathrm{T} - \mu \mathrm{R} ds \cos d\phi + \mathrm{R} ds \sin d\phi = 0 \quad . \quad (1)$$

$$(\mathrm{T} + d\mathrm{T}) \sin d\psi - \mathrm{R} ds \cos d\phi - \mu \mathrm{R} ds \sin d\phi = 0 \quad . \quad (2)$$

T and T + dT being the tensions at P and Q.

Now if we neglect higher powers of $d\psi$ than the first, we may put

$$\cos d\psi = 1, \ \cos d\phi = 1, \ \sin d\psi = d\psi, \ \sin d\phi = d\phi \quad . \quad (3)$$

Also the last term in each of the equations (1) and (2) is the product of two infinitely small quantities, and may be neglected, since it is infinitely smaller than the other terms.

With the substitutions given in (3) the equations (1) and (2) become

$$d\mathrm{T} - \mu \mathrm{R} ds = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$(\mathrm{T} + d\mathrm{T}) d\psi - \mathrm{R} ds = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

CHAPTER VIII

STRINGS AND CHAINS UNDER GRAVITY

187. In this chapter we shall deal with perfectly flexible strings and chains. A string is perfectly flexible when it offers no resistance to bending at any point. A string or chain approaches this ideal very nearly, particularly when the curvature is everywhere small. A string offers some resistance to bending at every point, and a chain will hardly bend at all under moderate forces except at the junctions of the links. But when the tension is much larger than the external force per foot, there is no appreciable error in treating a string as perfectly flexible. Also a chain whose links are short compared with the radius of curvature of the curve which it assumes, behaves very much like an ideal perfectly flexible string.

188. Uniform Catenary.—When a string is suspended with its ends attached to two points not in the same vertical line, the curve assumed by the string is called a *catenary*. If the string is of uniform density, that is, if its mass per unit length is constant, the catenary is called the *common* or *uniform* catenary. It is possible to make a chain hang in any one of an infinite number of types of curve by properly choosing the density of the curve, but only a few of these types bear the name of catenary.

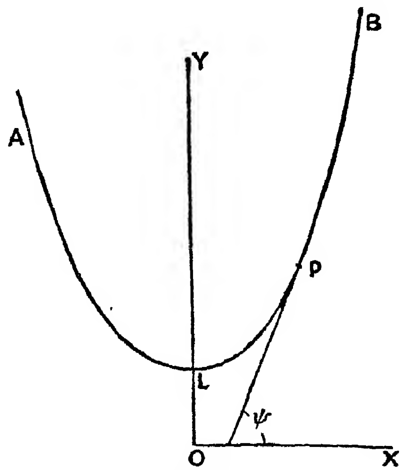


Fig. 89.

We will now find the equation of the uniform catenary. Let ALB denote the string of which L is the lowest point. The axis of y is taken as the vertical through L, and the axis of x will be fixed according to convenience. We are going to consider the equilibrium of the portion LP. Let the co-ordinates of P be x and y, and let the length of the arc LP be s. If w is the weight of unit length, the weight of LP is ws. The equilibrium of LP would clearly not be affected if this portion became rigid and the same forces acted on it. Consequently, the

forces acting on it satisfy the conditions of equilibrium of a rigid body. Denoting the tensions at L and P by T_0 and T, we get, by resolving vertically and horizontally,

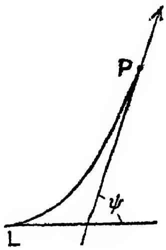


FIG. 90.

$$T \sin \psi = ws \quad . \quad . \quad . \quad . \quad (1)$$

$$T \cos \psi = T_0 \quad . \quad . \quad . \quad . \quad (2)$$

Dividing (1) by (2)

$$\tan \psi = \frac{w}{T_0} s \quad . \quad . \quad . \quad . \quad (3)$$

Writing c for $\frac{T_0}{w}$, this last equation can be written

$$s = c \tan \psi \quad . \quad . \quad . \quad . \quad (4)$$

This is the intrinsic equation, and from it we shall deduce the cartesian equation.

In any curve $\qquad \dfrac{dy}{d\psi} = \dfrac{dy}{ds} \cdot \dfrac{ds}{d\psi} = \sin \psi \dfrac{ds}{d\psi}$

But from (4) $\qquad \dfrac{ds}{d\psi} = c \sec^2 \psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$

Therefore $\qquad \dfrac{dy}{d\psi} = c \sin \psi \sec^2 \psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$

whence $\qquad y = c \sec \psi + A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$

Now let us take the axis of x at a distance c below L, so as to make the constant A zero. Then

$$y = c \sec \psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

We might now get x in terms of ψ by a similar method, and then find the relation between x and y by eliminating ψ from the expressions for x and y. But the following method is, in this case, rather simpler :—

$$\frac{dy}{dx} = \tan \psi$$

$$= \sqrt{(\sec^2 \psi - 1)}$$

$$= \sqrt{\left(\frac{y^2}{c^2} - 1\right)} \text{ by (8)}$$

Therefore $\qquad \dfrac{dx}{dy} = \dfrac{c}{\sqrt{(y^2 - c^2)}}$

from which $\qquad x = c \cosh^{-1} \dfrac{y}{c} + B \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$

But the constant B must be zero, because, by our choice of axes, $x = 0$ when $y = c$. Hence

$$y = c \cosh \frac{x}{c} = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right) \quad . \quad . \quad . \quad . \quad (10)$$

which is the required cartesian equation.

189. Some Properties of the Catenary.

(*a*) Equation (2) gives

$$T = T_0 \sec \psi$$

But $T_0 = cw$ and $y = c \sec \psi$. Hence

$$T = wy \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Thus the tension at any point is equal to the weight of a portion of the string which would extend in a vertical line from that point to the axis of x. An equivalent result was proved in Art. 152, Example 5.

(*b*) Since $$s = c \tan \psi$$

and $$y = c \sec \psi$$

it follows that $$y^2 - s^2 = c^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

(*c*) Again, using (12) and (10)

$$\begin{aligned} s^2 &= y^2 - c^2 \\ &= c^2 \cosh^2 \frac{x}{c} - c^2 \\ &= c^2 \sinh^2 \frac{x}{c} \end{aligned}$$

whence $$s = c \sinh \frac{x}{c} = \frac{c}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right) \quad . \quad . \quad . \quad . \quad (13)$$

It should be pointed out that s must be regarded as negative when x is negative. Moreover, ψ is negative in the same region, so that both equations (4) and (13) agree in giving a negative value to s when x is negative.

(*d*) From (10) $$\frac{dy}{dx} = \sinh \frac{x}{c}$$

Thus $$\tan \psi = \sinh \frac{x}{c} = \frac{s}{c} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

This result also follows from equations (4) and (13).

190. A uniform chain is suspended from two points in the same horizontal plane, and the inclination of the catenary at the points of suspension are known. To find the equation to the curve and its total length.

Let the horizontal distance between the points of suspension be $2a$, and let the value of $\tan \psi$ at one of the points of suspension be t. We will denote the value of s at the point of suspension where the inclination is given by s_1. The value of x at this point is a.

From (14), Art. 189,

$$\sinh \frac{a}{c} = t$$

that is
$$e^{\frac{a}{c}} - e^{-\frac{a}{c}} = 2t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

We have to find c from this equation. It is a quadratic in $e^{\frac{a}{c}}$. Thus, on multiplying by $e^{\frac{a}{c}}$, we get

$$\left(e^{\frac{a}{c}}\right)^2 - 2te^{\frac{a}{c}} - 1 = 0$$

whence
$$e^{\frac{a}{c}} = t + \sqrt{t^2 + 1}$$

the negative root being omitted since $e^{\frac{a}{c}}$ cannot be negative.

Hence
$$c = \frac{a}{\log_e (t + \sqrt{t^2 + 1})} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Thus c is found because a and t are given.
Now the equation is

$$y = c \cosh \frac{x}{c} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

If y_1 denotes the value of y at one point of suspension

$$y_1 = c \cosh \frac{a}{c}$$

This gives y_1, and therefore the position of the axis of x. Now the positions of the two axes are known, and (17) gives the complete equation to the curve.

Also by equation (4)

$$s_1 = c \, . \, t$$
$$= \frac{at}{\log_e (t + \sqrt{t^2 + 1})} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

and the total length of the chain is $2s_1$.

191. A uniform chain of length $2l$ has its ends attached to two points, A and B. To find the equation to the catenary in terms of l and the vertical and horizontal distances of B from A.

Let the equation be

$$y = c \cosh \frac{x}{c} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

where c is at present unknown and also the position of the origin unknown.

Let the co-ordinates of A be x, y, and those of B $(x + 2a)$, $(y + 2b)$, where a and b are known but x and y are unknown. Also let σ denote the value of s at A.

The quantities a, b, and l, are given, and our first task is to find c. By equation (19) we have

$$y = c \cosh \frac{x}{c} \quad \ldots\ldots\ldots \quad (20)$$

and by (13)

$$\sigma = c \sinh \frac{x}{c} \quad \ldots\ldots\ldots \quad (21)$$

Again, by (19)

$$\begin{aligned} y + 2b &= c \cosh \frac{x + 2a}{c} \\ &= c\left(\cosh \frac{x}{c} \cosh \frac{2a}{c} + \sinh \frac{x}{c} \sinh \frac{2a}{c}\right) \\ &= y \cosh \frac{2a}{c} + \sigma \sinh \frac{2a}{c}. \quad \ldots\ldots\ldots \quad (22) \end{aligned}$$

By (13) again

$$\begin{aligned} \sigma + 2l &= c \sinh \frac{x + 2a}{c} \\ &= c\left(\sinh \frac{x}{c} \cosh \frac{2a}{c} + \cosh \frac{x}{c} \sinh \frac{2a}{c}\right) \\ &= \sigma \cosh \frac{2a}{c} + y \sinh \frac{2a}{c}. \quad \ldots\ldots\ldots \quad (23) \end{aligned}$$

From (22) and (23)

$$\begin{aligned} 4l^2 - 4b^2 &= (y^2 - \sigma^2)\left\{\sinh^2 \frac{2a}{c} - \left(\cosh \frac{2a}{c} - 1\right)^2\right\} \\ &= c^2\left(\sinh^2 \frac{2a}{c} - \cosh^2 \frac{2a}{c} + 2 \cosh \frac{2a}{c} - 1\right) \end{aligned}$$

because $y^2 - \sigma^2 = c^2$ by (12)

But $\cosh^2 \frac{2a}{c} - \sinh^2 \frac{2a}{c} = 1$

Hence

$$4l^2 - 4b^2 = c^2\left(2 \cosh \frac{2a}{c} - 2\right) \quad \ldots \quad (24)$$

This is the equation to determine c. To simplify this equation, put z for $\frac{a}{c}$. Then

$$\begin{aligned} 2(l^2 - b^2) &= \frac{a^2}{z^2} (\cosh 2z - 1) \\ &= 2\frac{a^2}{z^2} \cdot \sinh^2 z \end{aligned}$$

Therefore

$$\sinh z = \frac{\sqrt{l^2 - b^2}}{a} z \quad \ldots\ldots\ldots \quad (25)$$

The easiest way to determine z is to plot the two curves

$$y = \sinh x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

$$y = \frac{\sqrt{l^2 - b^2}}{a}x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

and the value of x at their point of intersection is the value of z given by (25). A table of values of sinh x will be needed to do this work expeditiously.

Suppose z is found from equation (25). A glance at the form of the curve for (26) will show that it can only meet the line (27) once in the region where x is positive. There is, therefore, only one positive root of (25), and when this is found c is known because $c = \frac{a}{z}$. We have now to find the position of the origin.

After eliminating σ from (22) and (23) we get

$$y\left\{\sinh^2 \frac{2a}{c} - \left(\cosh \frac{2a}{c} - 1\right)^2\right\} = 2\left\{l \sinh \frac{2a}{c} - b\left(\cosh \frac{2a}{c} - 1\right)\right\}$$

or
$$2y\left(\cosh \frac{2a}{c} - 1\right) = 2\left\{l \sinh \frac{2a}{c} - b\left(\cosh \frac{2a}{c} - 1\right)\right\}$$

or
$$y = l\frac{\sinh 2z}{\cosh 2z - 1} - b$$

$$= l\frac{\cosh z}{\sinh z} - b \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

Since z is now known, this gives the value of y at A, and therefore the position of the x-axis is determined. The position of the y-axis can be determined by finding σ, and then using (21) to calculate x.

Thus, from (22) and (23), in the same way as y has been found, we get

$$\sigma = b\frac{\cosh z}{\sinh z} - l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

Then from (21) $\quad x = c \sinh^{-1} \frac{\sigma}{c}$

$$= c \log_e \frac{\sigma + \sqrt{(\sigma^2 + c^2)}}{c} \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

Thus x is determined, and therefore the position of the y-axis is known. The problem is now completely solved.

Numerical Example.—Suppose $a = 5$, $l = 12$, $b = 8$. Then the equation for z is

$$\sinh z = \frac{\sqrt{12^2 - 8^2}}{5}z = \frac{4\sqrt{5}}{5}z = 1{\cdot}789z$$

The value of z satisfying this equation is

$$z = 1{\cdot}976$$

Hence $$c = \frac{a}{z} = \frac{5}{1{\cdot}976} = 2{\cdot}530$$

$$\sigma = 8\frac{\cosh 1{\cdot}976}{\sinh 1{\cdot}976} - 12 = 8 \times 1{\cdot}0392 - 12 = -3{\cdot}686$$

$$x = 2{\cdot}530 \log_e \frac{-3{\cdot}686 + \sqrt{(3{\cdot}686^2 + 2{\cdot}530^2)}}{2{\cdot}530}$$
$$= (2{\cdot}530)(2{\cdot}303)\log_{10}(0{\cdot}310)$$
$$= -2{\cdot}96$$
$$y = 12 \times 1{\cdot}0392 - 8 = 4{\cdot}470$$

Thus the x-axis is at a distance $4{\cdot}470$ below the lower point of suspension, and the y-axis is between the two points of suspension and at a distance $2{\cdot}96$ from the lower point. The equation to the curve is

$$y = 2{\cdot}530 \cosh \frac{x}{2{\cdot}530}$$

192. To express x in terms of s and the height above the lowest point of a uniform catenary.

Let z denote the height of any point P above the lowest point of the catenary; x and y are the co-ordinates of P, and s is the length of the arc from the lowest point to P. Then

$$z = y - c = c\left(\cosh\frac{x}{c} - 1\right) = 2c\sinh^2\frac{x}{2c} \quad (31)$$

Also $$s = c\sinh\frac{x}{c} = 2c\sinh\frac{x}{2c}\cosh\frac{x}{2c} \quad . \quad . \quad . \quad (32)$$

Therefore $$\frac{z}{s} = \tanh\frac{x}{2c} = \frac{e^{\frac{x}{c}} - 1}{e^{\frac{x}{c}} + 1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

whence $$\frac{x}{c} = \log_e\left(\frac{1 + \frac{z}{s}}{1 - \frac{z}{s}}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

Also $$(z + c)^2 - s^2 = y^2 - s^2 = c^2$$

and therefore $$c = \frac{s^2 - z^2}{2z} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

Substituting this value of c in (34), we get

$$x = \frac{s^2 - z^2}{2z} \cdot \log_e\left(\frac{1 + \frac{z}{s}}{1 - \frac{z}{s}}\right)$$

$$= \frac{s^2}{2z}\cdot\left(1 - \frac{z^2}{s^2}\right)\cdot 2\left(\frac{z}{s} + \frac{1}{3}\cdot\frac{z^3}{s^3} + \frac{1}{5}\cdot\frac{z^5}{s^5} + \frac{1}{7}\cdot\frac{z^7}{s^7} + \ldots\right)$$

$$= s\left(1 - \frac{2}{1\cdot 3}\cdot\frac{z^2}{s^2} - \frac{2}{3\cdot 5}\cdot\frac{z^4}{s^4} - \frac{2}{5\cdot 7}\cdot\frac{z^6}{s^6} - \ldots\right) \quad . \quad . \quad . \quad (36)$$

If z is very small compared with s, as in the case of a telegraph wire, or a measuring chain stretched across a river or a hollow, we get a good approximation by omitting all powers of z beyond the second. Then

$$x = s - \frac{2}{3}\cdot\frac{z^2}{s} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

Thus, if the two ends of a measuring chain of length l are at the same level, the horizontal distance a between these ends is

$$a = l - \frac{8}{3}\cdot\frac{z^2}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

obtained by putting $\frac{a}{2}$ and $\frac{l}{2}$ for x and s in (7). z is the droop of the middle of the chain below the level of the ends.

If x is known, and an approximate value of s is wanted, we may write (37) thus

$$s = x + \frac{2}{3}\cdot\frac{z^2}{s}$$

Now s is nearly equal to x, and we may put x for s in the second term of this equation, since we shall only introduce a small error in a small term. Then

$$s = x + \frac{2}{3}\cdot\frac{z^2}{x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

193. Parabolic Catenary.—Suppose a chain is loaded so that the weight supported by a portion of the chain between any two points is proportional to the horizontal distance between verticals through those points. This is approximately the condition of a chain supporting a bridge whose weight per foot is constant. The ends of the chain are attached to two pillars, and the bridge is attached by vertical rods to different points of the chain.

Let the lowest point of the chain be taken as origin, and let w be the weight per horizontal foot. Then the weight supported by the piece of chain from the origin to the point (x, y) is wx. Hence, by the same method as for a uniform chain,

$$T \sin\psi = wx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$T \cos\psi = T_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Dividing (1) by (2),

$$\tan\psi = \frac{w}{T_0}x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

that is,
$$\frac{dy}{dx} = \frac{w}{T_0}x$$

Hence
$$y = \frac{1}{2}\cdot\frac{w}{T_0}x^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The constant of integration which should appear in (4) is omitted, because it is zero by our choice of the position of the origin.

Squaring (1) and (2), adding, and then taking square roots,

$$T = \sqrt{(T_0^2 + w^2x^2)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

By using (4) to express T_0 in terms of x and y, this can be written

$$T = \sqrt{\left(\frac{1}{4}\cdot\frac{w^2x^4}{y^2} + w^2x^2\right)}$$

$$= \frac{wx}{y}\sqrt{(\tfrac{1}{4}x^2 + y^2)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

194. Chain under any Load.—A chain is suspended from two points and acted on by no forces but its own weight and the reactions at the supports. To find equations for determining the form of the curve and the tension in the chain, the weight per unit length of chain being constant or variable.

Let W denote the weight of the chain from the lowest point L to any point P, whose co-ordinates are x and y.

By considering the equilibrium of the portion LP we get, in just the same way as in Art. 188,

$$T \sin \psi = W \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$T \cos \psi = T_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Dividing (1) by (2),

$$\frac{dy}{dx} = \tan \psi = \frac{W}{T_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In this equation W will, of course, be a variable quantity. In general, it can be expressed as a function of x or s. For the uniform catenary $W = ws$, and for the parabolic catenary $W = wx$. When W is expressed in terms of x or s equation (3) gives a differential equation for the curve in which the chain hangs.

If w is the weight per unit length of chain, constant or variable, we may write (3) in the form

$$\tan \psi = \frac{1}{T_0}\int w\,ds \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If, however, the shape of the curve is given and the density w is required, we must differentiate (4) with respect to s. Then

$$\sec^2 \psi \frac{d\psi}{ds} = \frac{w}{T_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

which gives w in terms of known quantities since the equation to the curve is known.

195. Other Forms of the Equations.—It is sometimes useful to have the equations of equilibrium of a chain in different forms from those obtained in (1) and (2) of Art. 194. All the different forms can, of course, be deduced from equations (1) and (2) by purely mathematical transformations; but we shall here obtain our equations by another method.

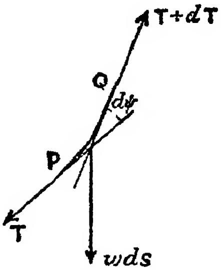


FIG. 91.

Consider the equilibrium of a small portion of the chain, PQ, of length ds. Resolving the forces along and perpendicular to the tangent at P,

$$(T + dT) \cos d\psi - T = wds \sin \psi$$
$$(T + dT) \sin d\psi = wds \cos \psi$$

where wds is the weight of the portion ds, T and T + dT the tensions at P and Q.

Retaining only first order quantities, these equations become

$$dT = wds \sin \psi = wdy$$
$$Td\psi = wds \cos \psi = wdx$$

dx and dy being the co-ordinates of Q relative to P. Hence

$$\frac{dT}{dy} = w \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$T\frac{d\psi}{dx} = w \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

These two equations may be used instead of (1) and (2) of Art. 194. Moreover, equation (6) is true even for a chain pressing against a smooth curve, because it was obtained by resolving along the tangent, and a normal pressure would not appear in this equation. If w is constant, equation (6) gives

$$T = wy + C$$

which agrees with the result obtained in example (5) in the chapter on virtual work.

196. Suppose two chains of equal lengths are suspended from two points A and B, and suppose their densities at a distance s from A (measured along the chains) are w and kw respectively, k being constant. Then the two chains will hang in the same curve.

For it is clear that, if T is the tension in the first chain, the equations (6) and (7) which will give the equation to the curve will only be multiplied all through by k if we put kw and kT for w and T respectively. Hence the differential equations with these substitutions will remain unaltered, and the only other conditions of the problem are the geometrical ones that the curves must both pass through the given points of suspension. Thus all the conditions for the curve of the second chain are the same as for the first if we put kT for T. This means that the chains will hang in the same curves, and their tensions will be in the ratio of their densities.

Thus two uniform chains of the same length will hang in the same curve between the same two points, whether their densities are equal or not.

Again, the previous argument can be modified so as to cover the parabolic catenary where the load is a uniform horizontal one. Whether a heavy or a light bridge is supported by a chain, provided the weight per horizontal foot is constant for each bridge, the chain will assume the same parabolic form.

197. We will now work out a few examples.

EXAMPLE (1).—*The equation to the curve in which a chain hangs is* $s = c \tan \psi$. *To find* w, *the weight of unit length of the chain.*

By (5), Art. 194,

$$w = T_0 \sec^2 \psi \frac{d\psi}{ds}$$

But here
$$\frac{ds}{d\psi} = c \sec^2 \psi$$

Hence
$$w = T_0 c = \text{a constant}$$

EXAMPLE (2).—*If the equation to the curve is* $y = kx^2$, *to find* w.

$$w = T_0 \sec^2 \psi \frac{d\psi}{ds}$$

But here
$$\tan \psi = \frac{dy}{dx} = 2kx$$

Therefore, differentiating again with respect to s

$$\sec^2 \psi \frac{d\psi}{ds} = 2k \frac{dx}{ds}$$

Hence
$$w = 2kT_0 \frac{dx}{ds} = 2kT_0 \cos \psi$$

We may write this result thus

$$w\,ds = 2kT_0\,dx$$

that is, the weight supported by ds is proportional to dx, its projection on the x-axis. This is the same as a uniform horizontal load, which has been considered in Art. 193.

EXAMPLE (3).—*Find* w *for a chain hanging in the form of a cycloid whose equation is* $s = 4a \sin \psi$.

$$s = 4a \sin \psi$$

Differentiating with respect to s

$$1 = 4a \cos \psi \frac{d\psi}{ds}$$

Hence
$$w = T_0 \sec^2 \psi \frac{d\psi}{ds}$$

$$= T_0 \frac{1}{4a} \sec^3 \psi$$

$$= T_0 \frac{16a^2}{(16a^2 - s^2)^{\frac{3}{2}}}$$

EXAMPLE (4).—*The load supported by a chain is* $ax^2 + b$ *per horizontal foot. To find the equation of the curve.*

The weight from the lowest point to x is

$$\int_0^x (ax^2 + b)dx = (\tfrac{1}{3}ax^3 + bx)$$

Putting this for W in equation (3), Art. 194, we get

$$T_0\frac{dy}{dx} = \tfrac{1}{3}ax^3 + bx$$

Therefore $$T_0 y = \tfrac{1}{12}ax^4 + \tfrac{1}{2}bx^2$$

No constant need be added if we take the origin at the lowest point of the curve.

198. Catenary of Uniform Strength.—If a rope be made so that the cross-section at every point is proportional to the tension it will have to bear, then it will be just as likely to break at one point as at another. If the only force on such a rope suspended from two points is its own weight, the curve assumed by the rope is called a *catenary of uniform strength.*

Let ρ be the weight of unit volume of the material, a the cross-section at any point. Then the volume of unit length in this neighbourhood is a, and therefore the weight w per unit length is ρa. Also $T = ka$. a is, of course, different at different points of the rope.

Substituting for w and T in equation (7), Art. 195, we get

$$ka\frac{d\psi}{dx} = \rho a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Therefore $$x = \frac{k}{\rho}\psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

the constant being zero if we measure x from the lowest point of the curve where $\psi = 0$.

Now $$\frac{dy}{dx} = \tan\psi = \tan\frac{\rho}{k}x \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Hence $$y = \frac{k}{\rho}\log_e\left(\sec\frac{\rho}{k}x\right) \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This is the cartesian equation to the curve.

Since ψ must lie between the limits $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$, equation (2) shows that x must lie between $-\frac{k\pi}{2\rho}$ and $+\frac{k\pi}{2\rho}$. Thus the maximum span for such a rope is $\frac{k\pi}{\rho}$.

Fig. 92 shows the form of the curve.

The quantity k is the tension across unit area of any normal section of the rope. If k be taken as the breaking tension of the material, the rope will be just on the point of breaking everywhere. We will now show that, with this value of k, the maximum span calculated for this

catenary is absolutely the greatest span it is possible to get with a rope of the given material.

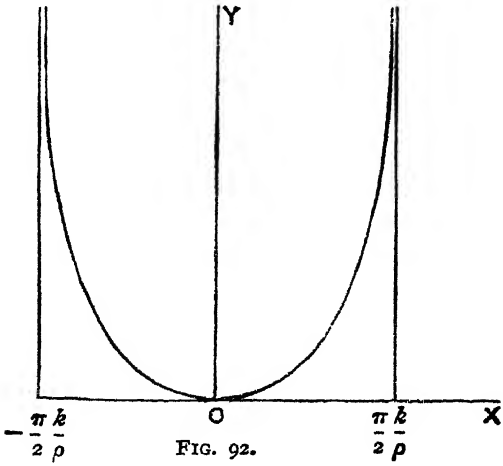


FIG. 92.

199. Equation (7), Art. 195, is

$$T\frac{d\psi}{dx} = w$$

Hence
$$\frac{dx}{d\psi} = \frac{T}{w} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

If $\frac{T}{w}$ is expressed in terms of ψ, the span of the catenary assumed by the rope is obtained by integrating (5) between the limits for ψ at the ends of the rope. Now ψ can never be greater than $\frac{\pi}{2}$, and never less than $-\frac{\pi}{2}$. Hence the maximum span for any rope is subject to the condition

$$\text{max. span} < \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{T}{w} d\psi$$

Now $w = \rho a$. Hence

$$\text{max. span} < \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{1}{\rho} \cdot \frac{T}{a} d\psi$$

$$i.e. < \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{1}{\rho} \cdot P d\psi$$

where P is the breaking tension for a section of unit area, for $\frac{T}{a}$ is the tension across unit area.

It follows that we get the greatest span in the catenary of uniform strength, where $\frac{1}{\rho} \cdot \frac{T}{a}$ has the greatest value it can possibly have, namely, $\frac{P}{\rho}$. And in this case the span is less than

$$\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{P}{\rho} d\psi, \text{ or less than } \pi\frac{P}{\rho}$$

Even in the catenary of uniform strength, ψ is only equal to $\frac{\pi}{2}$ when $y = \infty$, so that it is not possible in practice to make ψ quite equal to $\frac{\pi}{2}$.

The breaking tension for a strong steel rope is 130,000 lbs. per square inch. Also $\rho = 480$ lbs. per cubic foot. Hence the extreme limit for a span by any steel rope is

$$\frac{\pi \times 130{,}000 \times 144}{480} \text{ feet} = 23{\cdot}2 \text{ miles}$$

If we take the tension per unit area as $\frac{1}{6}$ of the breaking tension, the extreme limit for the span is $\frac{1}{6}$th of the above value, that is, nearly 4 miles.

If the extreme values of ψ are large, the droop of the curve at the middle will be very large. Suppose, then, the extreme values of ψ are $\pm\frac{\pi}{6}$. Then, since $x = \frac{k}{\rho}\psi$, the span will be $\frac{k}{\rho} \cdot \frac{\pi}{3}$. For the steel rope considered above, with a factor of safety six, the span is $\frac{4}{3}$ miles, and the droop at the middle is $\frac{k}{\rho} \log_e \sec \frac{\pi}{6}$, which is about 300 yards.

EXAMPLES ON CHAPTER VIII.

1. A uniform chain has a mass 2 lbs. per foot length, and the catenary in which it hangs has the equation

$$y = 10 \cosh \frac{x}{10}$$

Find, by using the tables, the tension and the vertical component of the tension at the point where $x = 25$ feet. *London B.Sc.*

[122·65 lbs., 121 lbs.]

2. A kite is flown with 600 feet of string from the hand to the kite, and a spring balance held in the hand shows a pull equal to the weight of 100 feet of the string, inclined at 30° to the horizontal. Find the vertical height of the kite above the hand. *London B.Sc.*

[556 feet.]

CHAPTER IX

MOMENTS OF INERTIA

200. Definition.—The product of the mass of a particle and the square of its distance from a given straight line is called the *moment of inertia* of the particle about that line. The moment of inertia of a system of particles about any line is the sum of the moments of inertia of the several particles.

The moment of inertia of a continuous body is represented by an integral. Thus suppose dm is the mass situated at a distance r from the line about which the moment of inertia is required, then the moment of inertia of the whole body is

$$\int r^2 dm$$

the limits of the integral being chosen so as to embrace the whole body.

If the density of the body is constant, dm is proportional to the volume occupied by this mass. If ρ is the density and dv the volume of dm, then $dm = \rho dv$, and the moment of inertia is

$$\rho \int r^2 dv$$

201. Moment of Inertia of an Area.—Sometimes we require the moment of inertia of an area in which the idea of mass is not involved. Suppose $d\text{A}$ is the area at a distance r from any straight line, then the moment of inertia of the area about that line is

$$\int r^2 d\text{A}$$

the integral extending over the whole area.

The moment of inertia of an area is really the same as the moment of inertia of the mass of a thin plate of uniform density having unit mass in unit area.

The student will see the necessity for moments of inertia in the chapters on rigid dynamics. In those chapters it will be seen that moments of inertia are as necessary as masses. In fact, moments of inertia occupy the same place in questions on the rotation of a rigid body as the mass does in questions on linear motion. And, moreover, the moment of inertia of an area will appear when we deal with the bending of beams in the next chapter.

The moment of inertia of several bodies about any line is the sum of the moments of inertia of the several bodies. For, by definition, the whole moment of inertia of the bodies is the sum of the moments of inertia of all the particles of all the bodies. But the sum of the

moments of inertia of the several bodies is also the sum of the moments of inertia of all the particles of all the bodies.

202. Radius of Gyration.—If I is the moment of inertia of a body about any line (or axis), and M its mass, the quantity $\sqrt{\frac{I}{M}}$ is called the *radius of gyration* of the body about that axis. If κ denotes this radius of gyration the relation between I, M, and κ, is given by

$$I = M\kappa^2$$

We do not usually need the radius of gyration itself. It is the square of this quantity that is of common occurrence. The quantity κ is clearly a length, because moments of inertia are the products of masses and squares of lengths.

If κ is the radius of gyration of several bodies about a given axis, and $M_1\kappa_1^2$, $M_2\kappa_2^2$, etc., are the moments of inertia of the bodies about the same axis, then, since the moment of inertia of all the bodies is the sum of the moments of inertia of the bodies,

$$(M_1 + M_2 + \ldots)\kappa^2 = M_1\kappa_1^2 + M_2\kappa_2^2 + \ldots$$

whence
$$\kappa^2 = \frac{M_1\kappa_1^2 + M_2\kappa_2^2 + \ldots}{M_1 + M_2 + \ldots}$$

203. We will now work out the moments of inertia of several bodies.

Example 1.—*The moment of inertia of a circular hoop of mass* M *and radius* a *about its axis of symmetry.*

Every particle of the hoop is at a distance a from the axis. Hence obviously

$$I = Ma^2$$

Example 2.—*The moment of inertia of a circular disc (regarded as an area) about its axis of symmetry.*

Let a be the radius of the disc. Let the disc be divided into thin rings or hoops by circles concentric with the disc. The area of the ring between the circles of radii r and $(r + dr)$ is $2\pi r dr$, and its moment of inertia about the axis of the disc is $r^2 . 2\pi r dr$. Hence the moment of inertia of the disc is

$$\int_0^a 2\pi r^3 dr = \tfrac{1}{2}\pi a^4 = \tfrac{1}{2}a^2 A$$

where A denotes the area of the disc.

If the disc be regarded as a mass, that is, as a uniform thin sheet of matter, the mass M would take the place of A in the result just obtained, because this result can be obtained in just the same way as the one above if we multiply all the areas by the mass in unit area, and this factor would turn A into M.

Example 3.—*Moment of inertia of any solid of revolution about its axis of symmetry.*

Let the solid be generated by the revolution of the curve $y = f(x)$ about the axis of x. Suppose the solid is divided into thin discs perpendicular to the axis of revolution. The thickness of a disc of radius y is dx, and its volume is $\pi y^2 dx$. If ρ is the density of the solid the

mass of the disc is $\pi\rho y^2dx$, and its moment of inertia, by the last example, is $(\pi\rho y^2dx)\frac{1}{2}y^2$. Hence the moment of inertia of the whole body is

$$\tfrac{1}{2}\pi\rho\int y^4dx = \tfrac{1}{2}\pi\rho\int\{f(x)\}^4dx$$

(*a*) For a cylinder of length l and radius a, the equation to the generating curve is $y = a$. Hence the moment of inertia is

$$\tfrac{1}{2}\pi\rho\int_0^l a^4dx = \tfrac{1}{2}\pi\rho a^4l = \mathrm{M} \times \frac{a^2}{2}$$

just as for a circular disc with the same mass and radius.

(*b*) The equation of the generating curve for a cone is $y = \frac{r}{h}x$, where r is the radius of the base and h the height The moment of inertia is therefore

$$\tfrac{1}{2}\pi\rho\int_0^h\frac{r^4}{h^4}\,.\,x^4dx = \mathrm{M} \times \tfrac{3}{10}r^2$$

(*c*) The equation of the generating curve for a sphere is $y^2 = a^2 - x^2$, a being the radius of the sphere. Hence the moment of inertia is

$$\tfrac{1}{2}\pi\rho\int_{-a}^{+a}(a^2 - x^2)^2dx = \mathrm{M} \times \tfrac{2}{5}a^2$$

(*d*) The moment of inertia of an infinitely thin spherical shell can be deduced from that of the sphere. If I denotes the moment of inertia of a sphere of radius a, $\mathrm{I} + d\mathrm{I}$ the moment of inertia of a sphere of radius $a + da$, then

$$\mathrm{I} = \tfrac{8}{15}\pi\rho a^5,\ d\mathrm{I} = \tfrac{8}{15}\pi\rho\,.\,5a^4da$$

But $d\mathrm{I}$ is the moment of inertia of the thin shell of thickness da and mass $4\pi a^2\rho da$. Hence, if M denotes the mass of the shell, its moment of inertia is $\mathrm{M} \times \frac{2}{3}a^2$.

EXAMPLE 4.—*The moment of inertia of a uniform thin rod of length l about an axis through the middle of the rod inclined at an angle θ to its length.*

Let O be the mid-point, P any other point, and Q a consecutive point on the rod. Let $\mathrm{OP} = x$, $\mathrm{PQ} = dx$, m = mass of unit length. Then mdx is the mass of dx, and $(mdx)(x \sin\theta)^2$ is its moment of inertia. Hence the moment of inertia of the whole rod is

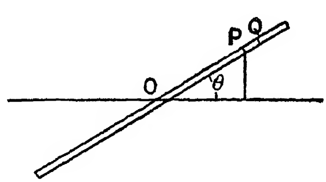


FIG. 93.

$$\int_{-\frac{l}{2}}^{+\frac{l}{2}} m\sin^2\theta\,.\,x^2dx = m\sin^2\theta\,.\,\tfrac{1}{12}l^3$$

$$= \mathrm{M}\frac{(l\sin\theta)^2}{12}$$

M being the total mass of the rod.

If $\theta = \frac{\pi}{2}$ the axis is perpendicular to the rod, and the moment of inertia is $M\frac{l^2}{12}$.

If $\theta = 0$ the axis is along the rod, and the moment of inertia is zero. This is obvious, because, all the mass being on the axis, all the r's in the expression $\int r^2 dm$ are zero.

EXAMPLE 5.—*The moment of inertia of a rectangle, whose sides are* a *and* b, *about an axis through the centre of gravity parallel to the sides of length* b.

This is easily proved to be the same as for a rod of equal mass and of length a, the axis being perpendicular to the rod. That is,

$$I = M\frac{a^2}{12}$$

If the moment of inertia of the area of the rectangle is wanted, the result is

$$I = A\frac{a^2}{12}$$

EXAMPLE 6.—*The moment of inertia of a triangular area about one side of the triangle. It is required to find the moment of inertia of the triangle* DBC *about the side* BC.

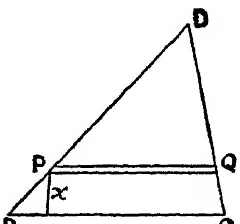


FIG. 94.

Let BC $= b$, and let the length of the perpendicular from D on BC be h.

Suppose the triangle is divided into thin strips parallel to BC. Let PQ be one of these strips, and let x be the distance of one side of the strip from BC, and $x + dx$ the distance of the other. The length of PQ is $\frac{h-x}{h}b$, and its area is therefore $\frac{h-x}{h}bdx$. Hence the moment of inertia of the triangle is

$$\int_0^h \frac{h-x}{h}bdx \,.\, x^2 = \frac{b}{h}(\tfrac{1}{3}h^4 - \tfrac{1}{4}h^4)$$

$$= \tfrac{1}{12}bh^3 = A\frac{h^2}{6}$$

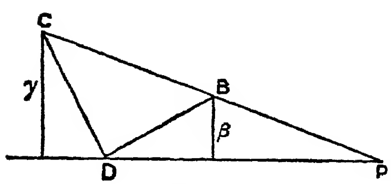


FIG. 95.

A being the area of the triangle.

In this case the square of the radius of gyration is $\frac{h^2}{6}$.

EXAMPLE 7.—*The moment of inertia of the area of a triangle about any axis in its plane through an angular point.*

Let DP be the axis, DBC the

triangle, and P the point where CB meets DP. Let the perpendiculars from B and C on the axis be β and γ, and let DP $= b$.

Now

$$\text{moment of inertia of DPC} = \text{moment of inertia of DPB} + \text{moment of inertia of DBC,}$$

that is,

$$\begin{aligned}\text{moment of inertia of DBC} &= \text{moment of inertia of DPC} - \text{moment of inertia of DPB}\\ &= \tfrac{1}{12}b\gamma^3 - \tfrac{1}{12}b\beta^3\end{aligned}$$

by the last example.

But
$$\begin{aligned}\text{area of DBC} &= \text{area of DPC} - \text{area of DPB}\\ &= \tfrac{1}{2}b\gamma - \tfrac{1}{2}b\beta = \tfrac{1}{2}b(\gamma - \beta)\end{aligned}$$

Hence

$$\begin{aligned}\text{moment of inertia of DBC} &= \tfrac{1}{12}b(\gamma^3 - \beta^3)\\ &= \tfrac{1}{2}b(\gamma - \beta) \times \frac{1}{6}\cdot\frac{\gamma^3 - \beta^3}{\gamma - \beta}\\ &= \text{A} \times \frac{1}{6}\cdot\frac{\gamma^3 - \beta^3}{\gamma - \beta},\\ \text{or} \quad &= \text{A}\frac{\gamma^2 + \gamma\beta + \beta^2}{6}\end{aligned}$$

where A denotes the area of the triangle.

If the axis is parallel to the side BC, β and γ are each equal to h the height of the triangle. Then the moment of inertia is A $\times \frac{h^2}{2}$.

If B and C are on opposite sides of DP, then β and γ must be considered to have different signs, so that the term $\gamma\beta$ is negative.

EXAMPLE 8.—*The moment of inertia of an ellipse about one of its principal axes.*

The equation of the ellipse referred to its principal axes is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

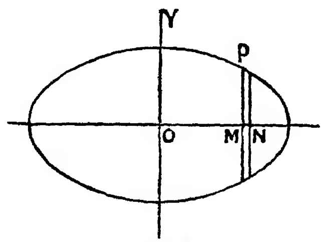


FIG. 96.

We will find the moment of inertia of the area about the y-axis.

Let the area be divided into thin strips parallel to the y-axis such as that shown in the figure. Let OM $= x$, MN $= dx$, MP $= y$. The area of the strip is $2ydx$, and its moment of inertia about OY is $2x^2ydx$. Hence the moment of inertia of the whole ellipse is

$$\int_{-a}^{+a} 2x^2y\,dx = \int_{-a}^{+a} 2x^2b\left(1 - \frac{x^2}{a^2}\right)^{\frac{1}{2}}dx$$

To evaluate this integral put $x = a \sin \phi$. Then $dx = a \cos \phi d\phi$, and the limits for ϕ are $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, corresponding to $-a$ and a for x. Thus

$$\int_{-a}^{+a} 2x^2 b\left(1 - \frac{x^2}{a^2}\right)^{\frac{1}{2}} dx = 2b \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} a^2 \sin^2 \phi \cos \phi \, . \, a \cos \phi d\phi$$

$$= 2ba^3 \, . \, \tfrac{1}{4} \, . \, \tfrac{1}{2} \, . \, \pi$$

$$= \mathrm{A} \times \frac{a^2}{4}$$

where $\mathrm{A} = \pi ab$, the area of the ellipse.

The moment of inertia of a circle about a diameter is included in the above result, for a circle is an ellipse with equal axes. If a is the radius, the moment of inertia is $\frac{1}{4}\mathrm{A}a^2$

204. If the position of a body be referred to three mutually rectangular axes OX, OY, OZ, the moments of inertia of the body about these axes are respectively

$$\int(y^2 + z^2)dm, \qquad \int(z^2 + x^2)dm, \qquad \int(x^2 + y^2)dm,$$

where dm is the mass situated at (x, y, z). For clearly $x^2 + y^2$ is the square of the distance of the point (x, y, z) from the z-axis, and a similar remark applies to the other quantities.

Suppose we are dealing with an area (or a plane body). If OX, OY be taken in the plane of the area, and OZ consequently perpendicular to it, the moment of inertia about the z-axis is thus

$$\int(x^2 + y^2)dm = \int x^2 dm + \int y^2 dm$$

Now for an area $\int x^2 dm$ is the moment of inertia of the area about the y-axis, and $\int y^2 dm$ is the moment of inertia about the x-axis. Thus if we denote the moments of inertia about the axes of x, y, and z, by I_x, I_y, and I_z, our equation can be written

$$\mathrm{I}_z = \mathrm{I}_x + \mathrm{I}_y$$

Thus we can find the moment of inertia of a plane body or area about a line perpendicular to its plane through any point O in that plane by finding the moment of inertia about any pair of perpendicular axes through O in the plane, and adding the results.

Thus the moments of inertia of a rectangle of sides a and b about axes through the centre of gravity parallel to the sides are $\frac{1}{12}\mathrm{A}a^2$ and $\frac{1}{12}\mathrm{A}b^2$. Hence the moment of inertia about an axis perpendicular to the plane through the centre of gravity is $\frac{1}{12}\mathrm{A}(a^2 + b^2)$.

Again, the moments of inertia of an ellipse about its principal axes of lengths $2a$ and $2b$ are $\frac{1}{4}\mathrm{A}b^2$ and $\frac{1}{4}\mathrm{A}a^2$. Hence the moment of inertia about an axis perpendicular to the plane through the centre of the ellipse is $\frac{1}{4}\mathrm{A}(a^2 + b^2)$. For a circle a and b are equal, and this last result becomes $\frac{1}{2}\mathrm{A}a^2$, which was the result obtained for a circular disc in Example 2, Art. 203.

205. To find a relation between the moments of inertia of any body about parallel axes.

Let G be the centre of gravity of the body, and O any other point. We shall find a relation between the moment of inertia of the body about a given axis through O and a parallel axis through G. The plane of the figure OGMP is perpendicular to the given axis through O. The body we are considering is not necessarily a plane body, but may be a solid body of any form extending on both sides of the plane of the figure.

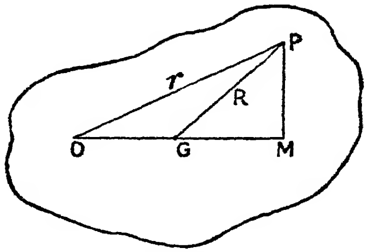


FIG. 97.

P is any point in the plane of the figure at a distance r from O and R from G. Let $GM = x$, $OG = h$. Now the whole body may be divided up into thin rods or columns parallel to the given axis, and therefore perpendicular to the plane OGP. Let dm be the mass of the column containing P. The moment of inertia of this column about O is

$$r^2 dm = (h^2 + R^2 + 2hx)dm$$

Hence the moment of inertia of the whole body

$$= \int(h^2 + R^2 + 2hx)dm$$

the limits of the integral being taken so as to include every column of the body.

Now h is the same for every column. Therefore

$$\int h^2 dm = h^2 \int dm = h^2 M$$

where M is the total mass.

Also $$\int 2hx dm = 2h\int x dm = 2h\bar{x}M$$

$\bar{x}$ being the x of the centre of gravity. But the origin for x is at the centre of gravity; consequently $\bar{x} = 0$, and therefore

$$\int 2hx dm = 0$$

Hence $$\int r^2 dm = h^2 M + \int R^2 dm$$

that is, the moment of inertia about the axis through O is equal to h^2M together with the moment of inertia about a parallel axis through G. In dealing with an area, the area A would take the place of M.

Thus, if we know the moment of inertia about any axis through the centre of gravity of a body, we can get the moment of inertia about any parallel axis by adding the product of the mass and the square of the distance of G from the new axis.

In some cases it is more convenient to calculate the moment of inertia of a body about an axis which does not pass through the centre of gravity. After having found this moment of inertia we can

get the moment of inertia about a parallel axis through G by subtracting the product of the mass and the square of the distance of G from the first axis.

We will apply the method to a few examples.

EXAMPLE 1.—*The moment of inertia of a thin rod about an axis perpendicular to the rod through one end.*

If M is the mass and l the length, the moment of inertia about a parallel axis through G is $\frac{1}{12}\text{M}l^2$. Hence the required moment of inertia is

$$\tfrac{1}{12}\text{M}l^2 + \text{M}\left(\frac{l}{2}\right)^2 = \tfrac{1}{3}\text{M}l^2$$

which can be easily verified by direct integration.

EXAMPLE 2.—*The moment of inertia of a triangular area about an axis through* G *parallel to one side, the perpendicular on that side from the opposite angular point being* h.

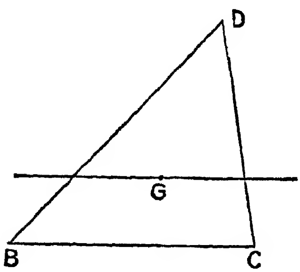


FIG. 98.

The moment of inertia about the parallel side we have already found (Art. 203, Ex. 6) to be $\frac{1}{6}\text{A}h^2$. Now the distance of G from the parallel side BC is $\frac{1}{3}h$. Hence the moment of inertia about the axis through G is

$$\tfrac{1}{6}\text{A}h^2 - (\tfrac{1}{3}h)^2\text{A} = \tfrac{1}{18}h^2\text{A}$$

The moment of inertia about an axis through D parallel to BC is now

$$\tfrac{1}{18}h^2\text{A} + (\tfrac{2}{3}h)^2\text{A} = \tfrac{1}{2}h^2\text{A}$$

which agrees with the result at the end of Ex. 7, Art. 203.

206. A very useful theorem concerning the moment of inertia of a triangle can now be proved. The theorem is :—

The moment of inertia of a triangle about any axis in its plane or parallel to it, is equal to the moment of inertia, about the same axis, of three particles situated at the mid-points of the sides of the triangle, each particle having a mass equal to one-third the mass (or area) of the triangle.

In Art. 203, Ex. 7, we found that the moment of inertia of a triangular area about an axis through an angular point is

$$\tfrac{1}{6}\text{A}(\beta^2 + \beta\gamma + \gamma^2)$$

Now the distances of the mid-points of the sides from the same axis are $\frac{\beta}{2}$, $\frac{\gamma}{2}$, and $\frac{\beta+\gamma}{2}$. Hence the moment of inertia about this axis of three equal particles of mass $\frac{1}{3}\text{A}$ at these points is

$$\begin{aligned} \text{I} &= \tfrac{1}{3}\text{A}\left\{\left(\frac{\beta}{2}\right)^2 + \left(\frac{\gamma}{2}\right)^2 + \left(\frac{\beta+\gamma}{2}\right)^2\right\} \\ &= \tfrac{1}{6}\text{A}(\beta^2 + \beta\gamma + \gamma^2) \end{aligned}$$

= the moment of inertia of the triangle about the same axis

Now the centre of gravity of these particles coincides with the centre of gravity of the triangle itself. Suppose d is the distance of this common centre of gravity from the axis. Then the moment of inertia of the triangle, or of the particles, about a parallel axis through G is $I - Ad^2$. Again, the moment of inertia of either system about a parallel axis at a distance h from G is obtained by adding Ah^2 to $I - Ad^2$. Hence the moment of inertia of the particles about any axis parallel to the original axis is the same as that of the triangle about the same axis. But the original axis can be parallel to any line in the plane of the triangle. Thus the theorem is proved.

This theorem gives the easiest method of finding the moment of inertia of any rectilinear figure about any line in its plane. It is only necessary to divide the figure into triangles and to replace each triangle by three particles, each having one-third the area of the triangle, at the mid-points of the sides of the triangle.

EXAMPLE.—*Find the moment of inertia of a regular hexagon about one side.*

Let A denote the area, a the length of a side. The hexagon can be divided into six equilateral triangles, each of area $\frac{1}{6}A$. If each triangle is replaced by three equal particles, these particles will have each a mass of $\frac{1}{18}A$. From a figure it will be seen that there is one particle on the axis, six particles at distance $\frac{\sqrt{3}}{4}a$ from the axis, four at distance $\frac{\sqrt{3}}{2}a$, six at distance $\frac{3\sqrt{3}}{4}a$, and one at distance $\sqrt{3}a$. Thus the moment of inertia is

$$\frac{1}{18}A\left(\frac{6 \cdot 3}{16}a^2 + \frac{4 \cdot 3}{4}a^2 + \frac{6 \cdot 27}{16}a^2 + 3a^2\right) = \frac{23}{24}Aa^2$$

207. Moment of inertia of a solid of revolution about an axis meeting the axis of revolution at right angles.

Let the axis about which the moment of inertia is required be taken as axis of y, and let $y = f(x)$ be the equation to the curve which, on being revolved about the x-axis, generates the surface of the solid. If P and Q are two neighbouring points on the curve whose abscissæ are x and $x + dx$ respectively, the volume of the circular disc generated by the area under PQ, is $\pi y^2 dx$, and its mass, taking ρ as density, is $\pi\rho y^2 dx$. The moment of inertia of this disc about OY is

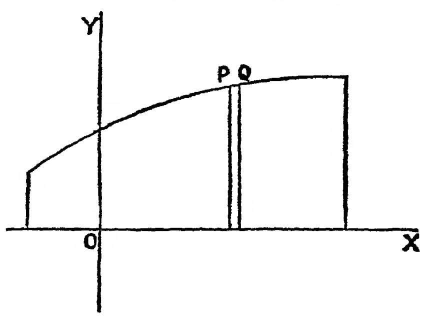


FIG. 99.

$$(\text{mass}) \times x^2 + (\text{mass}) \times \frac{y^2}{4}$$

the first term corresponding to Mh^2 of Art. 205, and the second being the moment of inertia of the disc about an axis parallel to OY through its own centre of gravity.

Putting $\pi\rho y^2 dx$ for the mass of the disc and integrating, we find that the moment of inertia of the whole solid is

$$\pi\rho\int_{x_1}^{x_2}\left(x^2+\frac{y^2}{4}\right)y^2 dx$$

x_1 and x_2 being the abscissæ of the bounding planes.

EXAMPLE 1.—*Suppose the body is a solid cylinder of uniform density, and* OY *passes through the centre of gravity.*

Let l be the length and a the radius of the ends. Then the equation of the generating curve is $y = a$, and the moment of inertia is therefore

$$I = \pi\rho\int_{-\frac{l}{2}}^{+\frac{l}{2}}\left(x^2+\frac{a^2}{4}\right)a^2 dx = \pi\rho a^2\left(\frac{l^3}{12}+l\,.\,\frac{a^2}{4}\right) = M\left(\frac{l^2}{12}+\frac{a^2}{4}\right)$$

M being the mass of the cylinder.

EXAMPLE 2.—*Solid cone about an axis through the vertex.*

Here the equation to the generating curve is $y=\frac{r}{h}x$, where r is the radius of the base, and h the height. Therefore

$$I = \pi\rho\int_0^h\left(x^2+\frac{r^2}{4h^2}x^2\right)\frac{r^2}{h^2}x^2 dx = \pi\rho\,.\,\frac{1}{5}h^5\left(1+\frac{r^2}{4h^2}\right)\frac{r^2}{h^2}$$

$$= M\times\tfrac{3}{5}\left(h^2+\frac{r^2}{4}\right)$$

208. Comparison of moments of inertia of a plane body about different axes in its plane, all passing through one point.

Let a pair of rectangular axes OX, OY, in the plane of the body have their origin at the fixed point, and suppose we know the values of $\int y^2 dm$, $\int x^2 dm$, and $\int xy dm$. We shall denote these quantities by A, B, and H respectively. Let OX′, OY′, be another pair of rectangular axes in the plane of the body, the angle XOX′ being θ. It is required to find the moment of inertia about OX′ in terms of A, B, H, and θ.

Let x', y', be the co-ordinates of a particle dm referred to the axes OX′, OY′, and (x, y) referred to OX, OY. Then

$$y' = y\cos\theta - x\sin\theta$$
$$x' = x\cos\theta + y\sin\theta$$

Hence the moment of inertia about OX′ is

$$\begin{aligned}\int y'^2 dm &= \int(y^2\cos^2\theta - 2xy\cos\theta\sin\theta + x^2\sin^2\theta)dm\\ &= \cos^2\theta\int y^2 dm - 2\sin\theta\cos\theta\int xy dm + \sin^2\theta\int x^2 dm\\ &= A\cos^2\theta - 2H\sin\theta\cos\theta + B\sin^2\theta\end{aligned}$$

By giving the proper value to θ we can use this result to find the moment of inertia about any axis through O in the plane of the body.

Again,

$$\begin{aligned}\int x'y'dm &= \int\{(y^2 - x^2)\sin\theta\cos\theta + xy(\cos^2\theta - \sin^2\theta)\}dm\\ &= (A - B)\sin\theta\cos\theta + H(\cos^2\theta - \sin^2\theta)\\ &= \tfrac{1}{2}(A - B)\sin 2\theta + H\cos 2\theta\end{aligned}$$

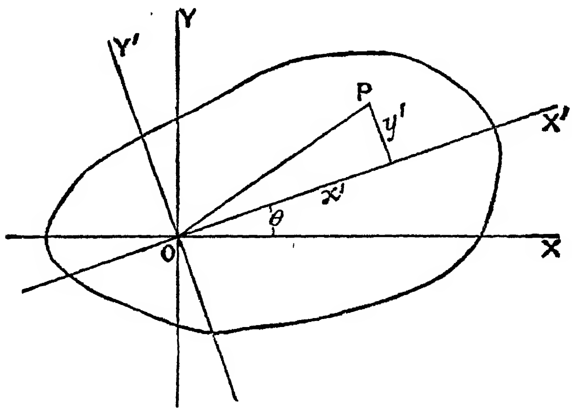


FIG. 100.

The quantity $\int x'y'dm$ is called the *product of inertia* of the body for the axes OX′, OY′.

For any given body there is a particular pair of axes, for which the product of inertia is zero. For, the product of inertia for the axes OX′, OY′, will be zero if

$$\tfrac{1}{2}(A - B)\sin 2\theta + H\cos 2\theta = 0$$

that is, if

$$\tan 2\theta = \frac{2H}{B - A}$$

Now for all possible values of A, B, and H, there is a value of θ between 0 and 90° which will satisfy this equation; for, as θ varies from 0 to 90°, tan 2θ varies from 0 to ∞, and then from $-\infty$ to 0. Thus there is certainly a pair of axes for which the product of inertia is zero.

The pair of axes through any point O, for which the product of inertia is zero, are called *principal axes* at that point.

If we had started with OX and OY as principal axes, then the moment of inertia about OX′ would have been

$$A\cos^2\theta + B\sin^2\theta$$

If A and B are equal, this moment of inertia is A for all values of θ. But if A and B are unequal and A greater than B, then A is the greatest and B the least moment of inertia for any axis through O in the plane of the body. For the moment of inertia about OX′ is

$$\begin{aligned} &A(1 - \sin^2\theta) + B\sin^2\theta \\ &= A - (A - B)\sin^2\theta \\ &= A - (\text{a positive quantity}) \end{aligned}$$

Also this same moment of inertia

$$\begin{aligned} &= A\cos^2\theta + B(1 - \cos^2\theta) \\ &= B + (A - B)\cos^2\theta \\ &= B + (\text{a positive quantity}) \end{aligned}$$

Thus the moment of inertia about OX′ cannot be greater than A nor less than B, and it assumes these extreme values when OX′ coincides with OX or OY.

It is easy to see that there is generally only one pair of principal axes through any point of the body, and these are the axes about which the moments of inertia are greatest and least for that point. If, however, $A = B$, all axes through the point are principal axes.

If a body has one symmetrical axis through any point, that must be one of the principal axes through that point. The other principal axis is, of course, perpendicular to this.

EXAMPLE 1.—*Rod about an inclined axis.*

The moment of inertia of a rod of length l about the line of the rod itself is zero, and about an axis perpendicular to the rod through its centre of gravity its moment of inertia is $\frac{1}{12}l^2M$. These are obviously the principal axes. Hence the moment of inertia about a line through the centre of gravity inclined at θ to the rod is

$$0 \cdot \cos^2\theta + \tfrac{1}{12}l^2M\sin^2\theta = \tfrac{1}{12}l^2M\sin^2\theta$$

which agrees with the result in Ex. 4, Art. 203.

EXAMPLE 2.—*The moment of inertia of a rectangular area, with sides* a *and* b, *about an axis through the mid-point of one of the sides of length* a, *making an angle* θ *with that side.*

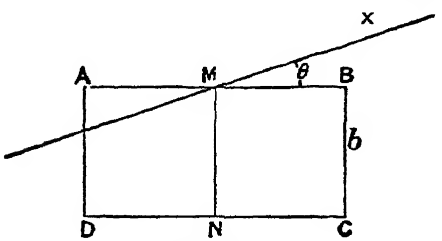


FIG. 101.

MX is the axis about which the moment of inertia is required. MN is the line joining the mid-points of opposite sides.

Since MN is a symmetrical axis, it is a principal axis through M. Now the moments of inertia about MN and AB are easily shown to be $\frac{1}{12}a^2A$ and $\frac{1}{3}b^2A$. Hence the moment of inertia about MX is

$$\tfrac{1}{12}a^2A\sin^2\theta + \tfrac{1}{3}b^2A\cos^2\theta = A(\tfrac{1}{12}a^2\sin^2\theta + \tfrac{1}{3}b^2\cos^2\theta)$$

EXAMPLE 3.—*The moment of inertia of an ellipse about any axis in its plane.*

Let the axis be at a distance h from the centre of the ellipse, and

let it make an angle θ with the major axis of the ellipse. The moment of inertia about a parallel axis through the centre of gravity is

$$\tfrac{1}{4}b^2\text{A}\cos^2\theta + \tfrac{1}{4}a^2\text{A}\sin^2\theta = \text{A}(\tfrac{1}{4}b^2\cos^2\theta + \tfrac{1}{4}a^2\sin^2\theta)$$

Hence the moment of inertia about the given axis is

$$\text{A}(\tfrac{1}{4}b^2\cos^2\theta + \tfrac{1}{4}a^2\sin^2\theta + h^2)$$

209. Comparison of Moments of Inertia of a Solid Body about Axes through One Point.—Corresponding to the relation proved in the last article between the moments of inertia of a plane body about axes in its plane all passing through one point, there is a similar relation between the moments of inertia of a solid body about axes passing through one point which are not restricted to lie in one plane. This relation we will now state without proof.

Through any point in a solid body there is a set of three mutually perpendicular axes, OX, OY, OZ, such that

$$\int yzdm = 0, \ \int zxdm = 0, \ \int xydm = 0$$

Let A, B, C, denote the moments of inertia of the body about OX, OY, OZ, respectively. Then the moment of inertia about a line through O which makes angles with the axes whose cosines are l, m, n, is

$$\text{A}l^2 + \text{B}m^2 + \text{C}n^2$$

The moments of inertia A, B, C, are called the *principal moments of inertia* at O, and the axes about which these are the moments of inertia are called the *principal axes of inertia* at O.

If a body has a symmetrical plane through any point, it is obvious that one principal axis is perpendicular to this plane, and the other pair are therefore in the plane.

If a body has a symmetrical axis through any point, that axis is a principal axis at that point, and any pair of rectangular axes through the point and perpendicular to the axis of symmetry can be taken as the other principal axes.

EXAMPLE 1.—*The moment of inertia of a solid cylinder, of length* b *and radius* r, *about an axis through its centre of gravity inclined at an angle* θ *with the axis of symmetry.*

Let the axis of z be taken along the axis of symmetry, and the axis of x in the plane of the axis of symmetry and the given axis. Then, from a figure,

$$l = \sin\theta, \quad m = 0, \quad n = \cos\theta$$

Also $\quad \text{A} = \text{B} = \tfrac{1}{4}\text{M}r^2 + \tfrac{1}{12}\text{M}b^2$

and $\quad \text{C} = \tfrac{1}{2}\text{M}r^2$

Therefore the required moment of inertia is

$$\text{I} = \text{M}(\tfrac{1}{4}r^2 + \tfrac{1}{12}b^2)\sin^2\theta + \text{M}\,.\,\tfrac{1}{2}r^2\cos^2\theta$$

EXAMPLE 2.—*Moment of inertia of a cone, with height* h *and radius of base* r, *about a line through the vertex inclined at* θ *to the axis.*

With co-ordinate axis in the same position as in Ex. 1,

$$A = B = M(\tfrac{3}{5}h^2 + \tfrac{3}{20}r^2)$$
$$C = M \,.\, \tfrac{3}{10}r^2$$

Therefore $$I = M(\tfrac{3}{5}h^2 + \tfrac{3}{20}r^2)\sin^2\theta + M \,.\, \tfrac{3}{10}r^2\cos^2\theta$$

210. One of the principal axes through any point of a plane body is perpendicular to the plane of the body on account of symmetry about that plane. Now suppose the moment of inertia is required about a line inclined to the plane at an angle θ. Let the perpendicular to the plane through the point of intersection of the plane and the given line be taken as axis of z. The other two principal axes at that point are in the plane, and are the same as those considered in Art. 208, where we took no account of the z-axis.

In this case

$$I = Al^2 + Bm^2 + C\sin^2\theta$$

But by Art. 204,

$$C = A + B$$

Hence $$I = A(l^2 + \sin^2\theta) + B(m^2 + \sin^2\theta)$$

It follows from this that two masses in the same plane which have the same principal axes at any point in the plane, and the same moments of inertia about those axes, will have the same moments of inertia about any axis through that point inclined to the plane.

For example, it was proved in Art. 206 that a triangle of mass M has the same moment of inertia about any axis in its plane as three particles each of mass $\frac{1}{3}$M placed at the mid-points of its sides.

Since the principal axes in their plane through any point are the axes of greatest and least moments of inertia, it follows that the triangle and the particles have the same principal axes. Hence, A, B, l, m, are the same for both systems. Therefore the two systems have equal moments of inertia about all axes.

211. Equimomental Systems.—If two bodies or systems of bodies have equal masses, the same centre of gravity, the same principal axes at the centre of gravity, and the same moments of inertia about those axes, then they have the same moments of inertia about all other axes.

Two such systems are called *equimomental systems*.

The proof is very simple. The moment of inertia about an axis through G whose direction-cosines are l, m, n, is, for either system,

$$Al^2 + Bm^2 + Cn^2$$

About an axis parallel to this at a distance h from the centre of gravity, the moment of inertia of either system is

$$Al^2 + Bm^2 + Cn^2 + Mh^2$$

Example.—The moments of inertia of a solid rectangular block with sides a, b, c, about its principal axes through its centre of gravity (which axes are clearly parallel to the sides), are

$$A = M\left(\frac{b^2}{12} + \frac{c^2}{12}\right),\ B = M\left(\frac{c^2}{12} + \frac{a^2}{12}\right),\ C = M\left(\frac{a^2}{12} + \frac{b^2}{12}\right)$$

CHAPTER X

ELASTICITY

213. Stress.—One of the fundamental assumptions in the statics of a rigid body is that two equal but opposite forces, acting in the same straight line but applied at different points, can balance each other. Now this implies that a rigid body can transmit through its substance a force applied at any point. Let us suppose, for the sake of illustration, that a rod is in equilibrium under the action of two equal pulls at the ends. It is clear that by some process, the ultimate nature of which we do not understand, the particles (or molecules) at which one pull is directly applied exert a pull on the next set of particles, and these again pull the next, and so on, the action applied at one end being handed on from particle to particle till it reaches the particles to which the force is applied at the other end. There must therefore be an attraction between neighbouring particles of the rod which resists their separation—provided the pull exerted is not too large.

If the forces at the ends of the rod had been directed towards, instead of away from, each other, there would have been a different kind of action between the particles. In place of an attraction there would have been a repulsion between the successive particles in the line of the rod.

Any mutual action between the neighbouring particles of a body is called a *stress*. The former of the above stresses is called a *tension*, and the latter, a *thrust*. Either of these two stresses may be regarded as the negative of the other, since, on changing the signs of the applied forces in any particular case, the tensions will become thrusts, and the thrusts tensions.

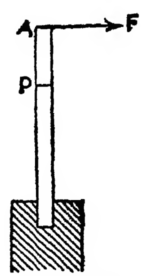


FIG. 102.

214. Shear Stress.—There is another kind of stress besides those we have just mentioned, and this we shall now consider.

Suppose a vertical rod is held firmly at one end so that its direction and position is fixed, and suppose a horizontal force F is applied at the other end A. Now let us consider the mutual actions of the parts of the rod on the upper and lower sides of a horizontal section through any point P of the rod. By resolving horizontally the forces acting on the portion AP of the rod, we see that there must be at P a horizontal force equal to F, but acting in the opposite direction. Thus each portion of the rod on opposite sides of the horizontal plane through P exerts on the other portion a force

which is in the plane of separation. This force is called a *shearing force*, or simply, a *shear*.

A shear is quite different from a thrust or tension, for these are forces perpendicular to the plane across which they are transmitted, whereas the shear is in that plane.

215. The three kinds of stresses usually co-exist in any solid body. It is only in exceptional cases that tension or thrust could exist without shear across some plane. As an example of the co-existence of tension and shear, we need only consider the action across an oblique section of a rod under no forces but two equal pulls at its ends. Since the resultant force across the section is along the rod, it follows that the force transmitted across the oblique section has a component in the plane of the section as well as one perpendicular to it. Thus there is a shear and a tension at the section.

A thrust between two portions of the same body is similar to a normal pressure between bodies in contact, and a shear is similar to friction. There is this difference, however, between shear and friction, that there can be no friction without a normal pressure, whereas shear can exist without any thrust or tension across the same plane. There is no force between two solid bodies which is similar to tension in a body.

216. Strain.—Until the present we have always assumed that a rigid body is one whose shape does not alter when the body is acted on by forces; that is, it was assumed that there could be no relative displacements of the particles of the body. This was, in fact, involved in the definition of a rigid body. But such bodies do not exist in nature. The smallest external force will alter the relative positions of the particles of an apparently rigid body, although the displacements may be so small that we cannot detect them without refined measurements. But every one is familiar with the effect produced on india-rubber or jelly by a pressure at one point, as well as with the effect produced on a rubber string by pulls at its ends. These bodies yield visibly to the thrusts or pulls applied to them. Now the same kind of yielding occurs when a pressure is applied to a block of iron, or a pull to an iron rod. But here the yielding is so small as to be generally unnoticeable. When the forces which deform a body are removed, the body recovers its original shape, if these forces have not been too large. This property which bodies possess of recovering their original shape after being deformed is called *elasticity*.

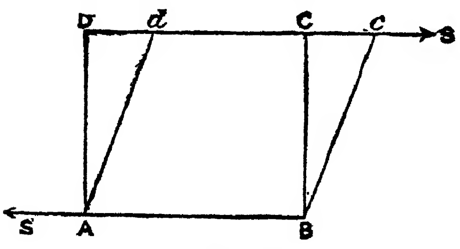


FIG. 103.

A body deformed by forces is said to be *strained*. To every stress there is a corresponding *strain*. A rod in thrust is shortened, and in tension it is lengthened. A shear acting on a cubical

block parallel to one of the edges distorts the sections which are parallel to the shear and perpendicular to its plane into parallelograms with unequal angles, as shown in Fig. 103.

When a rod whose unstrained length is l has its length increased by s, the measure of the strain is $\frac{s}{l}$. If the rod shortens, the contraction must be regarded as a negative extension, so that the strain will be negative.

The shear strain of the body shown in Fig. 103 is $\frac{Dd}{AD}$, and the corresponding stress acts in planes perpendicular to AD. Since Dd is small compared with AD, the radian measure of the angle DAd may be taken as the shear strain.

217. Hooke's Law.—This law states that the tension in an elastic body is proportional to the extension produced. Careful experiments show that the law can be extended to all kinds of stress and strain. The generalized law is:—

The stress producing any strain in a body is proportional to the strain. That is,

$$\frac{\text{stress}}{\text{strain}} = k \text{ (a constant)}$$

If the stress be taken as the force on unit area, the constant k is called the *modulus* of the substance for the particular kind of stress and strain we are dealing with. In the case of a tension or thrust the modulus is called *Young's modulus*, and in the case of shear it is called the *modulus of rigidity*. We shall denote these moduli by E and n respectively.

There is a limit for every substance beyond which Hooke's law is not true. If the stress is increased after this elastic limit is reached, the strain increases more for a given increase of stress than before the limit was reached. In addition, the body does not even approximately recover its original shape after the forces are removed. The strain left after the forces are removed is called *permanent set*. There is a certain maximum stress which cannot be exceeded without breakage, and this is called the *breaking stress*.

218. Elastic Strings.—If a string of length l and cross-section a, subject to a tension T, has an extension s, the tension on unit area of a normal section is $\frac{T}{a}$, and the strain is $\frac{s}{l}$. Hence, by Hooke's law

$$\frac{T}{a} = E\frac{s}{l} \quad \quad (1)$$

For the same string under different tensions a and l are constant, and therefore the above equation can be written

$$T = ks \quad \quad (2)$$

k being constant for the particular string only, while E is constant for all strings of the same material.

EXAMPLE 1.—*An elastic string of negligible weight and length* l *is attached to two points in the same horizontal plane, the string being just tight, but unstretched. When a weight* W *is attached to the middle point the two halves of the string are inclined at* θ *to the horizontal. To find* W *in terms of* θ.

The extension of the string is $l(\sec\theta - 1)$, and the strain therefore $(\sec\theta - 1)$. Hence

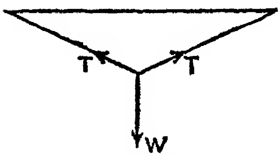


FIG. 104.

$$T = Ea(\sec\theta - 1)$$

But, for equilibrium of the attached weight,

$$\begin{aligned} W &= 2T\sin\theta \\ &= 2Ea(\tan\theta - \sin\theta) \end{aligned}$$

which is the required relation.

EXAMPLE 2.—*If* θ *is small in the last problem, to find an approximate value of* θ, *assuming* W *to be known.*

Expressing $\tan\theta$ and $\sin\theta$ in powers of θ, we have

$$\tan\theta = \theta + \frac{\theta^3}{3}, \qquad \sin\theta = \theta - \frac{\theta^3}{6} \text{ approximately}$$

Hence
$$W = 2Ea\frac{\theta^3}{2} = Ea\theta^3$$

Therefore
$$\theta = \sqrt[3]{\frac{W}{Ea}}$$

EXAMPLE 3.—AB *and* CD *are two elastic strings similar in all respects. The ends* A *and* C *are attached to two points at the same level and at a distance* b *apart. The lower ends are attached to a rod* AMB *of length* b, *which has its middle point* M *fixed by a hinge so that the rod can turn in the vertical plane containing* A *and* C. *The hinge at* M *is at a distance* l *vertically below the middle point of* AC. *If the natural length of each string is* a, *to find the extension when a weight* W *is attached at* D, *assuming that the shortened string does not become slack and that the strings only make small angles with the vertical.*

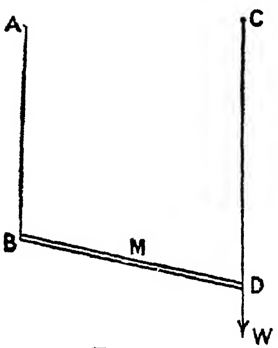


FIG. 105.

Let the increase in the length of CD due to W be y; then AB will decrease by approximately the same amount. If T_1, T_2, are the tensions, we have

$$T_1 = k(l + y - a), \quad T_2 = k(l - y - a)$$

Now taking moments about M for the equilibrium of the rod,

$$T_1 \cdot \frac{b}{2} - T_2 \cdot \frac{b}{2} - W \cdot \frac{b}{2} = 0.$$

therefore $$W = T_1 - T_2 = 2ky$$

Hence $$y = \frac{W}{2k}$$

and $$T_1 = k(l - a) + \frac{W}{2}, \quad T_2 = k(l - a) - \frac{W}{2}.$$

EXAMPLE 4.—ABCD *is a framework made of four equal rods of negligible weight joined by smooth hinges. Two exactly similar elastic strings join opposite corners of the quadrilateral. The framework is suspended by the hinge at* A, *and a weight* W *is attached to* C. *To find the new lengths of the strings, assuming that the shortened one does not become slack.*

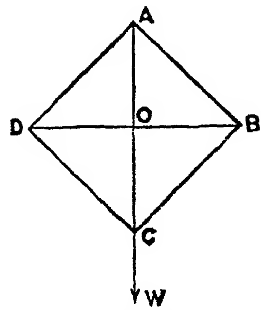


FIG. 106.

Let l denote the length of each string when the framework is in the form of a square, and let a denote the natural length of each string. Also let $l + y$ and $l - z$ denote the new lengths of the strings with the weight W attached. Then

$$OA^2 + OD^2 = DA^2$$

that is, $$\tfrac{1}{4}(l + y)^2 + \tfrac{1}{4}(l - z)^2 = \frac{l^2}{2}$$

Since y and z are small, we neglect the squares of these quantities and get

$$\tfrac{1}{2}ly - \tfrac{1}{2}lz = 0$$

or $$y = z$$

Now the tensions in AC and BD are $k(l + y - a)$ and $k(l - y - a)$ respectively. We shall call these T_1 and T_2.

To get a relation between the tensions and W we can use the principle of virtual work. If AC increases by an amount dy, then BD increases by an amount $d(l - z) = -dz = -dy$. Thus the work done by the tensions in a small variation of y is $-T_1dy + T_2dy$. Thus the equation of virtual work is

$$-T_1dy + T_2dy + Wdy = 0$$

Hence $$W = T_1 - T_2 = 2ky$$

Thus $$y = \frac{W}{2k}$$

and the new lengths of the strings are therefore

$$l + \frac{W}{2k} \text{ and } l - \frac{W}{2k}.$$

219. Work done in stretching an elastic string by a tension which, at any instant, is the same all along the string.

Let T be kx when the extension is x. Then the work done in a further extension dx is

$$Tdx = kxdx$$

Hence, the work done in altering the extension from x_0 to x_1 is

$$\int_{x_0}^{x_1} kxdx = \tfrac{1}{2}k(x_1^2 - x_0^2)$$
$$= \tfrac{1}{2}k(x_1 + x_0)(x_1 - x_0)$$
$$= \tfrac{1}{2}(T_1 + T_0)(x_1 - x_0)$$

which is the product of the arithmetic mean of the tensions at the beginning and end of the stretching and the increase in length.

If x_0, the original extension, was zero, the work done is $\frac{1}{2}T_1x_1$.

The above expression for work is the value of the work done by the stretching force, not by the tension in the string itself. The work done by the tension on the body which stretches the string is the negative of the result obtained.

If a body of weight W is attached to the end of an elastic string, the work done by this weight in descending x feet is Wx. The work done by the tension in the string on the body is $-\frac{1}{2}Tx$ in extending the string x beyond its natural length, where T is the tension at the end. Now if the body was released in the position where the extension was zero, it is shown in dynamics that it will come to rest again when the total work done on it is zero, that is, when $Wx - \frac{1}{2}Tx = 0$, or when $T = 2W$. Thus the body does not come to rest in the equilibrium position where $T = W$, but swings past it just as a pendulum bob does when it is released from any position except the equilibrium position.

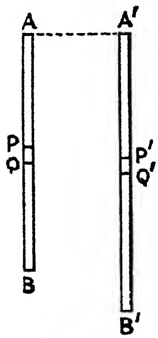


FIG. 107.

220. To find the extension of an elastic string under a continuous load distributed along its length.

Here the tension is not uniform, and consequently the strain varies along the length of the string. We shall therefore be obliged to consider the strain at each point of the string.

Let AB represent the unstretched string, A′B′ the stretched string. Let P, Q be two points on the unstretched string, P′,Q′, the corresponding points on the stretched string. Let $AP = x$, $A'P' = x + s$. Thus s denotes the extension of AP. Now let $PQ = dx$, then PQ′ will be $d(x + s) = dx + ds$. Then the strain in P′Q′ is $\frac{ds}{dx}$, since ds is the increase in length and the original length was dx.

If T denotes the tension at P, a the cross-section of the string, E Young's modulus,

$$T = Ea\frac{ds}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let w be the load per foot in the neighbourhood of P′. Then the load on P′Q′ is wdx, and this is balanced by the excess of tension at P over that at Q. Thus, $T + dT$ being the tension at Q,

$$T - (T + dT) = wdx;$$

that is,

$$\frac{dT}{dx} = -w \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In this equation w may, of course, be a function of x. By integrating equations (2) and (1) we can find the tension and extension at any point if we are given the conditions necessary to fix the constants of integration.

As an example take the case of a string under its own weight, w being constant, together with a load W at the end. Here

$$\frac{dT}{dx} = -w$$

Therefore
$$T = -wx + C \quad \ldots\ldots\ldots \quad (3)$$

or
$$Ea\frac{ds}{dx} = -wx + C$$

Whence
$$Eas = -\tfrac{1}{2}wx^2 + Cx + D \quad \ldots\ldots \quad (4)$$

To find the constants, we have the conditions that $T = W$ where $x = l$, and $s = 0$ where $x = 0$. These give

$$W = -wl + C$$
$$0 = D$$

Thus we get finally, on putting these constants in (3) and (4),

$$T = w(l - x) + W$$
$$Eas = \tfrac{1}{2}w(2lx - x^2) + Wx$$

The extension where $x = l$ is

$$s = \frac{1}{Ea}(\tfrac{1}{2}wl^2 + Wl)$$

221. Potential Energy of a Stretched String.—The potential energy of a stretched string is the work that could be done by the string in resuming its natural length. It may be regarded as energy stored up in the string which is available for doing work. This energy is the same as the work done by the stretching forces, provided the stretching takes place slowly; or it is the negative of the work done by the string against the stretching forces as the string is stretched from its natural length to its final state.

Suppose that when a string is being stretched the tension at a distance x from one end is T. Let s denote the extension at this point as in the last article. Now let us consider the work done by the tensions at the ends of a portion of natural length dx as it is extended to $dx + ds$. Since the length is small the tension will be, at any instant, nearly the same all along the portion. If T is the final tension in this element, the work done in stretching it is therefore (by Art. 219) $\frac{1}{2}Tds$, and the work done by the tensions themselves is the negative of this, namely, $-\frac{1}{2}Tds$. Hence the work done by the tensions as the string is stretched

$$-\tfrac{1}{2}\int T ds = -\tfrac{1}{2}\int_0^l T\frac{ds}{dx}dx$$

l being the natural length of the string.

The potential energy is the work which would be done by the string in contracting to the natural length again, and is therefore

$$+\tfrac{1}{2}\int \mathrm{T}\,ds = \tfrac{1}{2}\int_0^l \mathrm{T}\frac{ds}{dx}dx = \tfrac{1}{2}\int_0^l \frac{1}{a\mathrm{E}}\mathrm{T}^2 dx$$

If the tension is uniform the potential energy is, of course, $\tfrac{1}{2}\mathrm{T}^2\dfrac{l}{a\mathrm{E}}$, the same as the work done in stretching. The potential energy of the heavy string under its own weight alone is, by Art. 220,

$$\frac{1}{2}\int_0^l \frac{1}{a\mathrm{E}}w^2(l-x)^2 dx = \frac{1}{6}\cdot\frac{w^2}{a\mathrm{E}}l^3$$

$$= \frac{1}{6}\cdot\frac{(wl)^2}{a\mathrm{E}}l = \frac{1}{6}\cdot\frac{\mathrm{W}^2 l}{a\mathrm{E}}$$

where W is the total weight of the string.

This potential energy is one-third of that of the same string under a uniform tension equal to the weight of the string.

222. Elastic Rods.—In the theory of elastic solids it is assumed that the thrust required to shorten a rod by a length s is the same in magnitude as the tension required to lengthen it by s. This assumption is justified by experiments. Therefore the formula we have used for elastic strings, namely,

$$\frac{\mathrm{T}}{a} = \mathrm{E}\frac{s}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

can be applied to rods either in tension or thrust, with the conditions that a thrust shall be regarded as a negative tension and a contraction as a negative extension, and the value of E is the same in both cases.

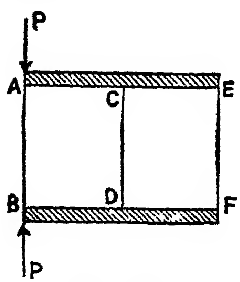


FIG. 108.

EXAMPLE 1.—AB, CD, EF, *are three equal and similar elastic rods whose ends are attached at equidistant points along two stout bars, which are supposed to remain always straight. At* A *and* B *two equal forces* P *are applied towards* B *and* A *respectively. To find the tensions or thrusts in the three rods.*

Suppose the rods AB, CD, EF, have extensions x, y, z, respectively. It is simpler to assume the rods are all extended, and let the signs of the extensions tell us whether they are extended or contracted. Since the connecting bars are always straight,

$$\mathrm{AB} + \mathrm{EF} = 2\mathrm{CD} \text{ always};$$

that is,

$$2l + x + z = 2l + 2y,$$

or

$$y = \tfrac{1}{2}(x+z) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The tensions in the rods are kx, ky, kz, k being the constant $\dfrac{\mathrm{E}a}{l}$.

Then considering the equilibrium of the bar AE, we have by moments about A,

$$ky + 2kz = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and by resolving parallel to the rods,

$$P = -k(x + y + z) \quad . \quad . \quad . \quad . \quad . \quad (4)$$

On solving equations (2), (3), and (4), we find

$$kx = -\tfrac{5}{6}P, \quad ky = -\tfrac{1}{3}P, \quad kz = \tfrac{1}{6}P$$

Thus there is a tension in EF and thrusts in the other two rods, as might have been expected.

It must be noticed that in this question we have assumed that there were no initial stresses in the rods. This would be quite true if the rods were exactly equal and the bars exactly straight.

EXAMPLE 2.—*If, instead of having* AE = 2AC *in the last question, we had had* AE = nAC *and the three rods parallel as before, what would then be the tensions and thrusts?*

By similar triangles it is easy to show that

$$EF - AB = n(CD - AB) \text{ always};$$

that is,

$$z - x = n(y - x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The equations corresponding to (3) and (4) of the last example are

$$ky + nkz = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$P = -k(x + y + z) \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Solving the equations (5), (6), (7) we get

$$\left.\begin{aligned} kx &= -\frac{n^2 + 1}{2(n^2 - n + 1)}P \\ ky &= -\frac{n^2 - n}{2(n^2 - n + 1)}P \\ kz &= \frac{n - 1}{2(n^2 - n + 1)}P \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

EXAMPLE 3.—*Six rods of the same material and thickness form the sides and diagonals of a square* ABCD, *and are joined together by smooth hinges at the corners. Four equal forces, each of magnitude* P, *are applied at the hinges, all the forces being perpendicular to* AB *and all acting outwards from the frame. Find the stress in each rod, assuming there were no initial stresses.*

From symmetry the stresses in the rods meeting at A will be equal to those in the corresponding rods at any other corner. We have, therefore, only to find the stresses at A.

Now, by considering the equilibrium of the hinge A we shall get two equations connecting these stresses. But we shall need a third equation. This third equation is a geometrical

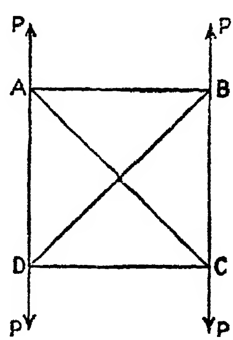


FIG. 109.

one, given by the fact that the lengths of the sides and the diagonals are not independent. The frame contains a redundant rod, and whenever this happens the length of one of the rods can be expressed in terms of the others. Every redundant rod gives one geometrical equation.

Let c denote the natural length of a side of the square; x, y, and z, the extensions of AB, AD, and AC. Then, since the figure will be a rectangle in the strained state,

$$(c+x)^2 + (c+y)^2 = (\sqrt{2}c+z)^2$$

Neglecting squares of x, y, and z, since these will be small in comparison with the first powers, this equation gives

$$x + y = \sqrt{2}z \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Let X, Y, Z, denote the tensions in AB, AD, AC. Then, if a denotes the area of the section of each rod,

$$X = Ea\frac{x}{c}, \quad Y = Ea\frac{y}{c}, \quad Z = Ea\frac{z}{\sqrt{2}c} \quad . \quad . \quad . \quad (10)$$

Substituting in (9) for X, Y, Z, we get

$$X + Y = 2Z \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Now resolving along AB and along AD for the equilibrium of the hinge at A,

$$X + \frac{1}{\sqrt{2}}Z = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

$$Y + \frac{1}{\sqrt{2}}Z = P \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Equations (11), (12), and (13), give

$$\left.\begin{aligned} X &= -\frac{\sqrt{2}-1}{2}P \\ Y &= \frac{3-\sqrt{2}}{2}P \\ Z &= \frac{2-\sqrt{2}}{2}P \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

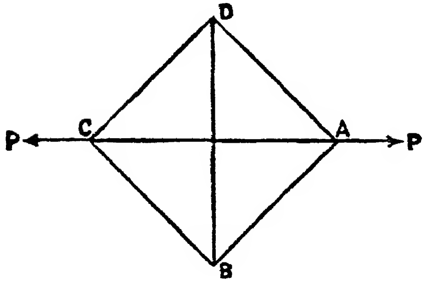


Fig. 110.

Thus there are tensions in AD and AC, and a thrust in AB.

Example 4.—*In the same framework as in the last example two balancing forces* P *are applied at* A *and* C, *acting outwards from the frame. Find the stresses in the rods.*

(*a*) *When the diagonal* BD *is strengthened so that its strain can be neglected.*

(*b*) *When the diagonal* BD *is left unstrengthened.*

(*a*) Let x denote the extension of the diagonal AC, and z the extension of each side. Then, since the diagonals remain perpendicular after strain,

$$\left(\frac{c}{\sqrt{2}}+\frac{x}{2}\right)^2+\left(\frac{c}{\sqrt{2}}\right)^2=(c+z)^2$$

that is, neglecting squares of x and z,

$$\frac{x}{\sqrt{2}}=2z \quad \text{. . . . . (15)}$$

Denoting the tensions in the diagonal and the sides by X and Z we have

$$X=Ea\frac{x}{\sqrt{2}c}, \quad Z=Ea\frac{z}{c} \quad \text{. . . . . (16)}$$

Equations (15) and (16) give

$$X=2Z \quad \text{. . . . . (17)}$$

Also, for the equilibrium of the hinge at A

$$P=X+\frac{2Z}{\sqrt{2}} \quad \text{. . . . . (18)}$$

From (17) and (18)

$$\left.\begin{aligned} Z&=\frac{2-\sqrt{2}}{2}P \\ X&=(2-\sqrt{2})P \end{aligned}\right\} \quad \text{. . . . . . . (19)}$$

The tension Y in the diagonal CD is given by considering the equilibrium of the hinge at D. Thus

$$Y=-\sqrt{2}Z=\frac{2-\sqrt{2}}{\sqrt{2}}P=(\sqrt{2}-1)P \quad \text{. . . (20)}$$

(*b*) Let the extensions of AC, BD, be x, y, and the extension of each side z. Then, because the diagonals are always perpendicular,

$$\left(\frac{c}{\sqrt{2}}+\frac{x}{2}\right)^2+\left(\frac{c}{\sqrt{2}}+\frac{y}{2}\right)^2=(c+z)^2$$

or

$$\frac{x}{\sqrt{2}}+\frac{y}{\sqrt{2}}=2z \quad \text{. . . . . . (21)}$$

Therefore

$$Ea\frac{x}{\sqrt{2}c}+Ea\frac{y}{\sqrt{2}c}=2Ea\frac{z}{c};$$

that is

$$X+Y=2Z \quad \text{. . . . . . (22)}$$

Also, by resolving along AC for the equilibrium of the hinge at A,

$$X+\sqrt{2}Z=P \quad \text{. . . . . (23)}$$

and resolving along BD for the equilibrium of the hinge at D,

$$Y+\sqrt{2}Z=0 \quad \text{. . . . . (24)}$$

Solving (22), (23) and (24), we get

$$\left.\begin{aligned} X &= \frac{\sqrt{2}}{2}P \\ Y &= -\frac{2-\sqrt{2}}{2}P \\ Z &= \frac{\sqrt{2}-1}{2}P \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (25)$$

Example 5.—ABCD *is a rectangular framework formed of rods* AC, AD, BC, BD, CD. *There are smooth hinges at the corners of the rectangle, and the hinges at* A *and* B *are fixed to two points in a vertical line. All the rods are of the same material and the same cross-section. To find the stresses in them due to a load* W *applied at* C, *the lower end of* CD.

Let the natural lengths of AD and DC be a and b. The length of each diagonal will be $\sqrt{a^2+b^2}$, and we shall denote this by d for brevity. Let the angle ADB be α. Now let the tensions in AD, AC, BD, BC, CD, be X, Y, Y′, X′, Z, respectively, and let the extensions be x, y, y', x', z.

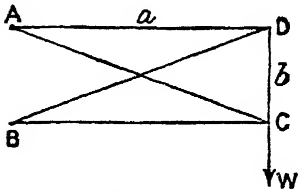


Fig. 111.

Now the problem can be resolved into two easier problems similar to each other. We can deal with the rods AD, DB, acted on by the force Z, the tension in CD. Then we shall deal with the rods AC, CB, acted on by an upward force at C, whose magnitude is Z − W. We shall then have to express the condition that the sum of the downward displacement of D and the upward displacement of C is the contraction in the rod CD caused by the thrust −Z. It is, of course, obvious beforehand that this contraction will turn out negative; that is, there will be an extension in DC.

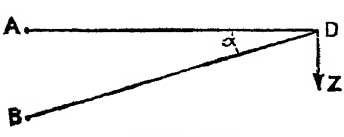


Fig. 112A.

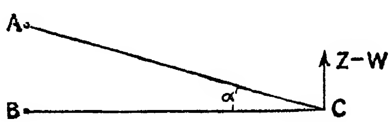


Fig. 112B.

For the equilibrium of the hinge at D we have the two statical equations, obtained by resolving horizontally and vertically,

$$X + Y'\cos\alpha = 0 \quad \ldots\ldots\ldots \quad (26)$$

$$Z + Y'\sin\alpha = 0 \quad \ldots\ldots\ldots \quad (27)$$

Now suppose that, after the strain, the angle DAB becomes θ. Then D descends $a\cos\theta$ approximately. We have now to find this quantity in terms of a, b, and the strains x and y'.

By trigonometry, from the triangle ABD after strain,

$$\cos\theta = \frac{(a+x)^2 + b^2 - (d+y')^2}{2(a+x)b}$$
$$= \frac{ax - dy'}{ab}$$

on neglecting squares and products of x and y'.

Thus D descends

$$\frac{ax - dy'}{b} \quad \ldots \ldots \ldots \quad (28)$$

Now turning to the other part of the frame, we get the two statical equations

$$X' + Y\cos\alpha = 0 \quad \ldots \ldots \ldots \quad (29)$$
$$(Z - W) + Y\sin\alpha = 0 \quad \ldots \ldots \ldots \quad (30)$$

Also C rises

$$\frac{ax' - dy}{b} \quad \ldots \ldots \ldots \quad (31)$$

Thus CD is shortened by

$$z = \frac{ax - dy' + ax' - dy}{b} \quad \ldots \ldots \quad (32)$$

Since the extension of each rod is proportional to the product of the tension in the rod and the length, equation (32) gives

$$-bZ = \frac{a^2X - d^2Y' + a^2X' - d^2Y}{b} \quad \ldots \ldots \quad (33)$$

Now by means of equations (26), (27), (29), (30), we can express all the other tensions in terms of Z. On substituting these values in (33), we get

$$-b^2Z = \frac{a^3 + d^3}{b}\cdot(2Z - W)$$

Therefore

$$Z = \frac{a^3 + d^3}{2a^3 + b^3 + 2d^3}W \quad \ldots \ldots \quad (34)$$

$$W - Z = \frac{a^3 + b^3 + d^3}{2a^3 + b^3 + 2d^3}W \quad \ldots \ldots \quad (35)$$

Also

$$\left.\begin{aligned} Y' &= -\frac{d}{b}Z, & X &= \frac{a}{b}Z, \\ Y &= +\frac{d}{b}(W - Z), & X' &= -\frac{a}{b}(W - Z) \end{aligned}\right\} \quad \ldots \quad (36)$$

Thus all the stresses are known.

In all the preceding examples we have assumed that there were no initial stresses in the rods. But it can easily be shown that, if there are initial stresses, all our equations would remain true if we substituted everywhere the *increases* in the tensions due to the applied forces instead of the whole tensions. For our geometrical equations clearly give the increases in the extensions of the rods, and these are proportional to the increases in the tensions. Moreover, every hinge is in

equilibrium before and after the application of the external forces; and consequently, by subtracting the corresponding equations expressing equilibrium in the two states, we shall obtain new equations differing from the equations of equilibrium when the initial stresses are zero only in having the excesses of the tensions due to the applied forces instead of the whole tensions.

223. Shearing of Short Beams.—Suppose a short beam is acted on by forces perpendicular to its length, and suppose that there is no appreciable bending; that is, suppose that there is no relative lengthening of the fibres of the beam in the direction of its length. Let y be the deviation, from the original central axis of the beam, of a point on the central axis at a distance x from one end (or from some fixed point on the beam). Then by definition $\frac{dy}{dx}$ is the shear strain. The shear stress per unit area of the section is therefore equal to $n\frac{dy}{dx}$, n being the modulus of rigidity.

EXAMPLE 1.—*For cast iron* $n = 6 \cdot 10^6$ *lbs. per square inch. To find the deflection due to shear at the end of a horizontal cast-iron beam of rectangular section, when a weight* W *lbs. is suspended at one end and the other end is firmly fixed in a wall. Length of beam standing out of wall 2 feet, depth 1 foot, breadth 6 inches. The weight of the beam itself may be neglected.*

Let y' inches be the end deflection. In this case the shear strain is constant, so that $\frac{dy}{dx} = \frac{y}{x}$. Thus the strain is $\frac{y'}{24}$, and the stress per square inch is $\frac{W}{72}$. Hence $\frac{W}{72} = \frac{y'}{24} \times 6 \cdot 10^6$

Therefore $$y' = \frac{W}{18 \times 10^6} \text{ inches}$$

If W is 20 tons = 44,800 lbs., then $y' = 0{\cdot}0025$ inch.

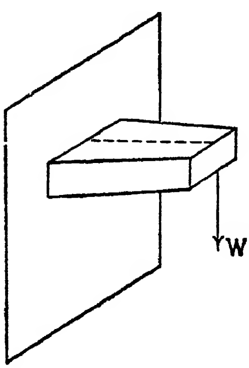


FIG. 113.

EXAMPLE 2.—*A beam is bounded by horizontal and vertical faces, and a horizontal section of the beam is a quadrilateral with two sides parallel. The thick end is built into a wall, and a weight* W *is suspended from the other end: to find the deflection due to shear.*

Let the widths of the horizontal section at the wall and at the free end be a and b respectively; and let l be the length of the beam, d its depth.

This example differs from the last one in that the shear stress on unit area of a normal section is different at different points of the beam. For the total shearing force at any section is constant, while the area is variable

Let y denote the deflection at distance x from the wall. The area of this section is

$$d\left\{a - (a - b)\frac{x}{l}\right\}$$

Hence the equation connecting stress and strain gives

$$n\frac{dy}{dx} = \frac{\mathrm{W}}{d\left\{a - (a - b)\frac{x}{l}\right\}}$$

Integrating, we get

$$ny = -\frac{\mathrm{W}l}{d(a - b)}\left[\log_e\left\{a - (a - b)\frac{x}{l}\right\} + \mathrm{C}\right]$$

To make $y = 0$ when $x = 0$, C must be $-\log_e a$

Then $$y = -\frac{\mathrm{W}l}{nd(a - b)}\log_e\frac{a - (a - b)\frac{x}{l}}{a}$$

and the deflection at the end, obtained by putting $x = l$, is

$$\frac{\mathrm{W}l}{nd(a - b)}\log_e\frac{a}{b}$$

224. Twisting of a Circular Rod.—Assuming that the lower end of a very short rod is fixed, we shall find what couple must be applied to the upper end to turn it through a small angle ϕ.

Let l denote the length of the rod, and let dA denote the sectional area of a very thin prismatic portion of the whole rod.

There is one line in the rod which is not displaced by the couple. This is denoted by CD in the figure.

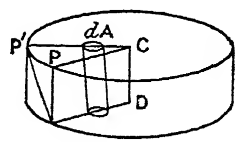


FIG. 114.

The line CP is strained into the position CP′ by the couple, so that PCP′ is the angle ϕ.

Let the distance of dA from C be denoted by r, and let the co-ordinates of dA referred to a pair of axes through C in the plane PCP′ be x, y. Then the shear strain of the thin column dA is $\frac{r\phi}{l}$; consequently the shear stress is $n\frac{r\phi}{l}$, perpendicular to CP′ (since ϕ is small). Thus the force on the upper dA is $\frac{nr\phi}{l}d\mathrm{A}$ perpendicular to CP′. This force has components $\frac{nx\phi}{l}d\mathrm{A}$ and $-\frac{ny\phi}{l}d\mathrm{A}$ parallel to the axes. Since the whole action on the upper face is known to be a couple, the vector sum of all the shear forces is zero. That is,

$$\frac{n\phi}{l}\int x\,d\mathrm{A} = 0, \text{ and } \frac{n\phi}{l}\int y\,d\mathrm{A} = 0$$

But if $\bar{x}, \bar{y}$, are the co-ordinates of the centre of gravity of the area A of the upper section of the prism, these equations mean

$$\frac{n\phi}{l}\bar{x}A = 0, \text{ and } \frac{n\phi}{l}\bar{y}A = 0$$

Thus the origin, namely C, is the centre of gravity of the area. This shows that the line which remains unmoved by a couple applied to a rod is the line joining the centres of gravity of the sections.

To find the moment of the couple, take moments about C of all the shear forces, and the moment of the couple is found to be

$$\frac{n\phi}{l}\int r^2 dA = \frac{n\phi}{l}I$$

where I denotes the moment of inertia of the area of the section about the axis of twist.

Since ϕ and l are small, let us now write them $d\phi$ and ds, so that $d\phi$ is the angle through which an infinitesimal portion ds of a prism is twisted. Then the couple on any section is

$$nI\frac{d\phi}{ds}$$

If the couple is constant throughout the length of the prism, as when couples are applied at the ends only, then $\frac{d\phi}{ds}$ will be constant also, and becomes again $\frac{\phi}{l}$, where ϕ and l are no longer small quantities, but are the total twist and the whole length of any prism.

For a hollow cylinder with inner and outer radii a and b the couple on each end which will produce a twist ϕ is

$$\frac{n\phi}{l}\cdot\tfrac{1}{2}\pi(b^4 - a^4)$$

The work done by the couple $\frac{n\phi}{l}I$ in twisting one end of the rod of length l through the angle $d\phi$, while the other end remains fixed, is

$$\frac{n\phi}{l}I\,d\phi$$

Consequently, the whole work done in twisting the end through ϕ_1 radians is

$$\int_0^{\phi_1}\frac{n}{l}I\phi\,d\phi = \frac{1}{2}\cdot\frac{n}{l}I\phi_1^2 = \tfrac{1}{2}nI\left(\frac{\phi_1}{l}\right)^2 l$$

$\frac{\phi_1}{l}$ being the final angle of twist per unit length.

This gives the potential energy of the twisted rod.

If a small piece ds of a rod is twisted through $d\phi$, we find the work

done in twisting this by putting ds and $d\phi$ for l and ϕ_1 above. Thus the work is

$$\tfrac{1}{2}n\mathrm{I}\left(\frac{d\phi}{ds}\right)^2 ds$$

If $\dfrac{d\phi}{ds}$ is not constant at every point of the rod, or if the rod is not straight, the last result shows that the work done in twisting the whole rod of length l is

$$\int_0^l \tfrac{1}{2}n\mathrm{I}\left(\frac{d\phi}{ds}\right)^2 ds$$

This, of course, includes the case where $\dfrac{d\phi}{ds}$ is constant, and therefore equal to $\dfrac{\phi}{l}$ for a straight rod.

The preceding is Coulomb's theory of torsion and was applied by him to prisms of any section. But St. Venant subsequently showed that this theory is inaccurate for any but rods which are solid or hollow circular cylinders. The assumption made in the preceding proof, namely, that the shear strain of an elemental prism is proportional to $r\phi$, is true only if the end faces of this prism retain their original directions. From the definition of shear strain it is clear that the shear strain of a thin short prism is proportional to the change of the inclination of the length of the prism to its end faces. Thus the whole of our proof depends for its validity on the assumption that the end faces of the short prism do not alter their directions; that is, our argument requires that all the particles that were originally in a plane section remain in one plane section when the rod is twisted. St. Venant has shown[1] that Coulomb's assumption is strictly true only for a circular rod or tube that has an axis of symmetry. For rods that have very compact sections, such as regular hexagons or octagons, Coulomb's theory is nearly true. But if the section is not nearly circular, if, for example, the section is a triangle or a long thin rectangle, then Coulomb's formula is very far from the truth. For a section in the form of a square the factor I in the preceding formula should be replaced by 0·844I, and for an equilateral triangle it should be replaced by 0·60I.

The results proved above are true for all uniform prisms if we replace I by a coefficient K, which is less than I for all prisms except circular cylinders.

224a. Stretching of a Spiral Spring.—Suppose the coils of a spiral spring are very nearly perpendicular to the axis of the spring—which is usually true—and suppose that the ends of the spring are bent so that the hooks, at which the pulls are applied, lie on the axis of the spring.

Let equal pulls T be applied at the hooks, and consider the action at any normal section P of the wire. The portion of the spring from P to either end is in equilibrium under the force T at that end, and the

[1] See Prescott's *Applied Elasticity*.

action at the section P. To balance T this action must be a shearing force equal and opposite to T, and a couple in the plane containing P and the axis of the spring, which plane is very nearly coincident with the normal section at P. Assuming the coincidence to be exact, the couple produces a twist in the wire at P about the central axis of the wire at that point. Thus each small portion of the wire is subject to a torsion similar to that considered in Art. 224.

Let $a=$ the radius of the coils, $r=$ the radius of the wire, $n=$ the rigidity-modulus, $d\phi=$ the angle of twist of a length ds of the wire, $l=$ the total length of wire in the coils. Then, since the couple is aT, we get

$$\mathrm{K}n\frac{d\phi}{ds}=a\mathrm{T}$$

Now, to find the extension of the wire due to T, it is easiest to use the principle of virtual work. The work done in twisting all the infinitesimal portions of the wire is, by Art. 224,

$$\int_0^l \tfrac{1}{2}n\mathrm{K}\left(\frac{d\phi}{ds}\right)^2 ds=\int_0^l \frac{a^2\mathrm{T}^2}{2n\mathrm{K}}ds=\frac{a^2\mathrm{T}^2 l}{2n\mathrm{K}}$$

If x is the total extension of the spring, the work done by the tensions T is

$$\int_0^x \mathrm{T}\,dx$$

Now, since the shear strain is very small, and therefore the work done by the shearing force T is a negligible quantity, we may equate these two expressions for the work done. Thus—

$$\int_0^x \mathrm{T}\,dx=\frac{a^2\mathrm{T}^2 l}{2n\mathrm{K}}$$

Differentiating with respect to x, we get

$$\mathrm{T}=\frac{a^2 l}{2n\mathrm{K}}2\mathrm{T}\frac{d\mathrm{T}}{dx};$$

whence
$$\frac{d\mathrm{T}}{dx}=\frac{n\mathrm{K}}{a^2 l},$$

and therefore
$$\mathrm{T}=\frac{n\mathrm{K}}{a^2 l}x$$

No constant need be added, since T is clearly zero when x is zero.

This gives T in terms of x, or x in terms of T, and we see that one is directly proportional to the other, just as for an elastic string.

If the points at which the equal forces T are applied do not lie on the axis of the spring we get a different relation between T and x.

Let us consider a single coil of the wire. Suppose the line of action of the tensions meets the plane of this coil (the coil being assumed to be in one plane) at a point H distant c from the centre of the coil. Let p denote the perpendicular distance of H from the tangent to the element ds. Then the torsion in ds is pT, and now we have

$$Kn\frac{d\phi}{ds}=pT$$

If θ is the angle which p makes with the line joining H to the centre of the coil, then

$$p = a + c\cos\theta$$

Writing $ad\theta$ for ds, we find that the work done in twisting this coil is

$$\int_0^{2\pi}\tfrac{1}{2}nK\left(\frac{d\phi}{ds}\right)^2 ad\theta = \frac{aT^2}{2nK}\int_0^{2\pi}p^2d\theta$$
$$= \frac{aT^2}{2nK}\int_0^{2\pi}(a^2 + 2ac\cos\theta + c^2\cos^2\theta)d\theta$$
$$= \frac{aT^2}{2nK}(2\pi a^2 + \pi c^2)$$

If the wire of the spring had unit mass in unit length, its moment of inertia about the line of the tensions, if the line makes a small angle with the axis of the spring, would be $2\pi a(a^2 + c^2)$. If we call this $2\pi a\kappa^2$, then the work done in twisting the coil may be written

$$2\pi a(a^2 + \kappa^2)\frac{T^2}{4nK}$$

To get the work done on all the coils we must sum these quantities for every coil, and it is clear that this sum is

$$l(a^2 + \kappa^2)\frac{T^2}{4nK}$$

where κ is the radius of gyration of the spring about the line of the tensions. This differs from the work when the tensions are axial only in having $(a^2 + \kappa^2)$ instead of $2a^2$. Since the rest of the work is exactly the same as before, it follows that

$$T = \frac{2nK}{(a^2 + \kappa^2)l}$$

225. Bending of Thin Rods or Beams.—Suppose a thin, naturally straight, uniform rod or beam AB is acted on by no forces but two equal and opposite couples at its ends. In order that any portion, AP, of the beam should be in equilibrium, there must be a couple acting on this portion at P to balance the couple at A. That is, the action at P of the part PB on the part AP is a couple exactly equal to the couple at B. Similarly, the action of AP on PB is a couple at P equal to the one at A

FIG. 115.

It is clear from the above that any portion of the beam bounded by normal sections is under the same kind of action as the whole beam, namely, a pair of opposing couples at the ends, the magnitudes of whose moments are the same as those applied at A and B. It follows, therefore, that all equal small elements of the beam will be similarly deformed. This deformation is a bending. Thus, every element will be bent into

a curve, and since the deformation is the same for every element, the whole beam must have the shape of a circular arc.

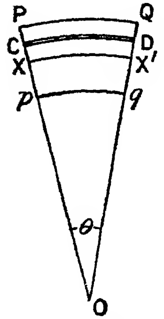


FIG 116A.

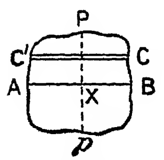


FIG. 116B.

226. We shall now calculate the couple required to bend a given uniform rod into an arc of a circle of radius R.

Let $PQqp$ be a portion of the bent rod. The sections Pp, Qq, which were originally normal plane sections, will be nearly plane after bending, and we shall assume they are plane, and that these planes meet in a line through O perpendicular to the plane of the paper, and are inclined at an angle θ.

The figure $PQqp$ is, of course, only a section of the beam in the plane of bending. Fig. 116B may represent an end view of the section at P.

Now it is clear that there is some plane of particles such as XX′, meeting the section at P in the line AB, whose length remains unaltered by the bending. We shall show that AB passes through the centre of gravity of the area of the section at P. This line AB is called the *neutral axis*.

Let CC′ denote a small strip of area dA parallel to AB. Then the bundle of fibres CD, meeting the section at P in CC′, has a new length, $(R + z)\theta$, where z is the distance of CC′ from AB. But the natural length of CD is the length of XX′, that is, $R\theta$. Hence the strain of CD is $\frac{z}{R}$. The strain of a bundle of fibres on the side of AB towards O will still be $\frac{z}{R}$ if we regard z as negative there.

The tension necessary to stretch CD is $E\frac{z}{R}dA$. Hence the total tension at the section through P is

$$\frac{E}{R}\int z\,dA \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

But since we know that the action at this section is a couple, the tension must be zero. That is,

$$\int z\,dA = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

But if $\bar{z}$ is the distance of the centre of gravity of the section from AB, we know that

$$\int z\,dA = \bar{z}\int dA \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Thus equations (2) and (3) tell us that $\bar{z} = 0$, that is, the line AB passes through the centre of gravity of the section.

The moment about AB of the tension on CC′ is

$$zE\,.\,\frac{z}{R}dA = \frac{E}{R}z^2dA \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The sum of the moments, M, of all the tensions across the section at P is therefore given by

$$M = \frac{E}{R}\int z^2 dA = \frac{EI}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where I denotes the moment of inertia of the area of the section about AXB, which is a line through the centre of gravity of the section and perpendicular to the plane of bending.

If X is the centre of gravity of the section the moment of the tensions about PXp is $\frac{E}{R}\int yzdA$, which is zero by hypothesis. It follows then that XB, XP are principal axes at X. This is a necessary condition that the rod should bend in the plane of the couple.

If the neutral axis does not pass through the centre of gravity of the section, there is a tension T across the section given by

$$T = \frac{E}{R}\int zdA = \frac{E}{R}\bar{z}A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

A being the area of the section, and $\bar{z}$ being measured from the neutral axis.

In this case we may regard the total action across the section as equivalent to a couple, together with the above tension acting at some point in the line through the centre of gravity parallel to the neutral axis AB. We will now find the moment of this couple.

Let M denote the moment. Now T may be assumed to act at some point in the line where $z = \bar{z}$. Hence, if we take moments about this line, of the stresses across the section at P, we shall get M, since T has no moment about the line.

The tension on dA is

$$\frac{E}{R}zdA$$

and the distance of the line of action of this tension from the line about which we are taking moments is $(z - \bar{z})$. Hence its moment is

$$\frac{E}{R}(z - \bar{z})zdA$$

Thus $$M = \frac{E}{R}\int(z - \bar{z})zdA = \frac{E}{R}\left(\int z^2 dA - \bar{z}\int zdA\right)$$

$$= \frac{E}{R}\{(I + \bar{z}^2A) - \bar{z}^2A\}$$

because $\int zdA = \bar{z}A$, and $\int z^2 dA = I + \bar{z}^2A$ by Art. 205

Hence $$M = \frac{EI}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

exactly as when there is no resultant tension across the section.

The quantity M, which is the moment, about an axis through the centre of gravity of a section perpendicular to the plane of bending, of the tensions across that section (thrusts being negative tensions), is called the *bending moment* at that section.

227. Beams with Transverse Loads.—We shall now consider the action across a section of a beam acted on by forces in one plane and perpendicular to the beam.

Let PB be a portion of the beam, P being any point in it and B one end. Let R, S, T, denote the forces on PB.

Since PB is in equilibrium, the action of the other portion of the beam on PB must balance R, S, and T. Now the forces R, S, T, are equivalent to a force at P perpendicular to the beam together with a couple. Hence the action of the other portion of the beam must be a force perpendicular to the beam and a couple, in order to keep the portion PB in equilibrium. The force is a *shearing force* and the couple is a *bending moment*, and we shall denote them by F and M respectively.

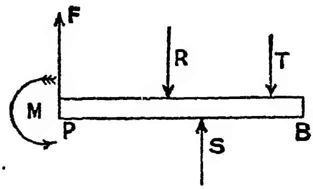


FIG. 117.

In some cases there is a couple acting at the end of the beam. But the argument which shows that a couple and a force at P is necessary for equilibrium would not be affected if a couple were applied at B. There would still be, in general, a bending moment and a shearing force at P.

For the sake of clearness we must have rules to fix when shearing force and bending moment are positive. The following are the rules we shall use in this book for horizontal beams.

The shearing force is considered positive when the left-hand portion exerts an upward force on the right-hand portion.

The bending moment is considered positive when the upper side of the beam is the convex side.

The direction of positive shear and bending moment are indicated in Fig. 117.

If we consider the external forces applied to PB to be positive when they act downwards and negative upwards, we get, by resolving perpendicular to the beam, for the equilibrium of PB,

$$F = \text{the sum of the external forces on PB} \quad . \quad . \quad (1)$$

Again, considering the moment of a downward force on PB as positive, we get, by taking moment about P

$$M = \text{the sum of the moments about P of the external forces on PB} \quad . \quad . \quad (2)$$

It must be observed that, since shearing force and bending moment are mutual actions between two portions of the beam, their directions on the two portions will be opposite. Thus, if AP is the other portion of the beam we have been dealing with, the directions of positive shearing force and bending moment are indicated in the figure. Equations similar to (1) and (2) would give F and M by considering the equili-

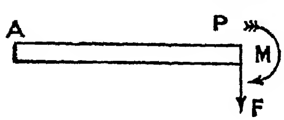


FIG. 118.

brium of AP. But in dealing with AP we should have to consider upward forces as positive in the equation for F. Downward forces would still, however, have a positive moment in the equation for M.

228. Shearing Force and Bending Moment for Isolated Loads.—The equation (1) in the last article shows that the shear between two consecutive applied forces is constant when the only forces acting on the beam are concentrated at single points. Thus, in Fig. 117, if we imagine P to travel along the beam towards B, the shear will be constant until the force R is reached. On passing R there is a sudden decrease in the shear by an amount R, and then the shear is again constant as far as S, on passing which it increases by S.

But there is no sudden alteration in the bending moment on passing a load as there is in the shear. For suppose x is the distance of R from P. The moment of R about P is Rx, and this quantity varies continuously with x; that is, there is no sudden change in the quantity as x approaches zero. After passing the load this term would not appear in the equation for M, as equation (2) shows. Thus the term due to the moment of R gradually approaches zero as P approaches R, and must be omitted (that is, must be taken as zero) after passing that force.

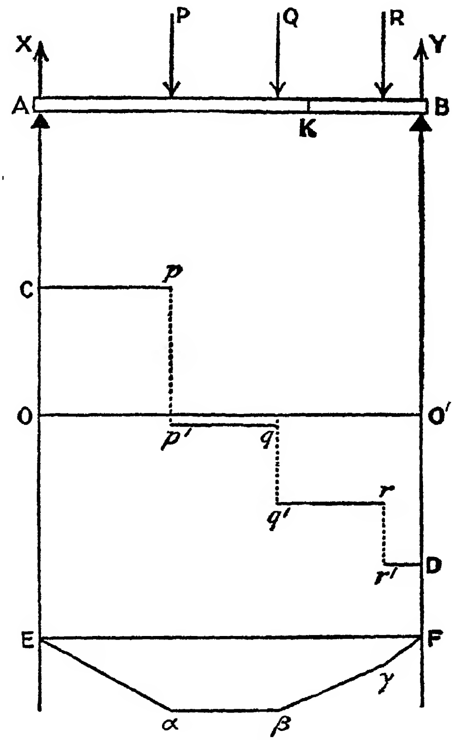


FIG. 119.

We will now represent shearing force and bending moment in two instances by means of diagrams. These diagrams give a very clear picture of the way in which shear and bending moment vary along the beam.

Figure 119 represents a beam AB supported on pegs at A and B, and acted on by the force P, Q, R, of given magnitude. Let the length of the beam be l, and let the distances of the lines of action of P, Q, R from A be a, b, c. Let x denote the distance of any point K from A.

By taking moments about A and B in turn, we find Y and X the reactions of the supporting pegs.

The ordinates of the discontinuous line C$pp'qq'rr'$D, measured from

OO′, represent the values of the shearing force at the points of the beam vertically above them.

The bending moment at K, obtained by taking moments about K for the equilibrium of AK, is

$$M = -xX + (x - a)P + (x - b)Q$$
$$= x(P + Q - X) - aP - bQ$$

Thus the curve for the bending moment between Q and R is a straight line whose slope is (P + Q − X), which is the negative of the shearing force in the same region. It is easy to see, therefore, that the bending moment curve is a series of sloping lines, the slope at each point being the negative of the shear at the corresponding point of the beam. Also, M is zero at each end and negative for downward applied forces at all intermediate points. The ordinate of the polygon $E\alpha\beta\gamma F$ at any point, measured from EF, gives the bending moment at the point of the beam vertically above it.

It is an easy matter to show that the polygon $E\alpha\beta\gamma F$ is a funicular polygon for the forces on the beam. This is left as an exercise for the student.

We take another example of a *cantilever*, that is, a beam built into a wall at one end and free at the other. In this case the bending moment is not zero at the point A where the beam leaves the wall.

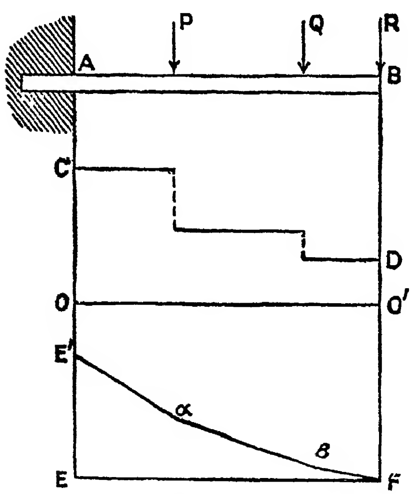


FIG. 120.

The ordinates of the broken line between C and D again represent the shear in the beam. The shear is obtained immediately from the given forces. Thus OC represents P + Q + R, the shear between A and the force P. The shear between P and Q is Q + R, and between Q and the end B it is R.

Now the bending moment curve can be constructed by starting at F and drawing three lines $F\beta$, $\beta\alpha$, $\alpha E'$, whose slopes are respectively R, (Q + R), (P + Q + R), the values of the shear in the corresponding regions.

We shall in future refer to an end of a beam built into a wall as a *fixed* end, and to an end supported on a peg as a *supported* end, and to an end not supported or held in any way as a *free* end. At a fixed end the direction, as well as the position, of the beam is fixed. At a supported end only the position is fixed.

229. Continuous Loads.—We will now prove some equations which are needed for finding the condition of a beam under a continuous load.

Let a line along the beam be taken as axis of x, and let w be the load per unit length at x. If w is constant there is a load w on each unit length of the beam, but if w is not constant the load on dx is wdx, and w must be regarded as a function of x. F and M will, of course, also be functions of x.

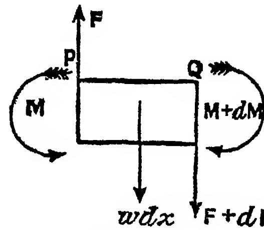


FIG. 121.

We shall consider the equilibrium of a small portion, PQ, of the beam. Let the x co-ordinates of P and Q be x and $x + dx$. The load on PQ is wdx. Resolving perpendicular to the beam for the equilibrium of PQ,

$$(F + dF) - F + wdx = 0$$

Whence
$$\frac{dF}{dx} = -w \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Also, taking moments about Q,

$$(M + dM) - M + Fdx - (wdx) . fdx = 0$$

where fdx denotes the distance of the line of action of wdx from Q; f is therefore a quantity less than unity.

Dividing through by dx after removing M and $-$M, we get

$$\frac{dM}{dx} = -F \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Equations (1) and (2) are the fundamental equations for continuous beams, or beams under continuous loads. If, in addition to the continuous load, there are finite loads at isolated points, there will be a sudden alteration in the shear on passing one of these loads, as was shown in Art. 228. There is no sudden alteration in the bending moment, but equation (2) shows that on passing a load there is a sudden change in the slope of the bending moment curve by an amount equal in magnitude to the change in the shear, that is, by an amount equal to the load at the point.

230. Deflection of Beams due to Bending.—Now let y be the downward displacement, due to bending of the beam, of a point of the beam whose abscissa is x. Then x and y are the co-ordinates of a line of particles in the beam which was originally straight. The radius of curvature, R, of this line is given by

$$\frac{1}{R} = \pm \frac{\dfrac{d^2y}{dx^2}}{\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In a bent beam $\frac{dy}{dx}$ is usually small compared with unity. If we ignore this term in the denominator of the expression for $\frac{1}{R}$ we get

$$\frac{1}{R} = \pm\frac{d^2y}{dx^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Now any small portion of the beam may be regarded as an arc of a circle whose radius is the radius of curvature of the beam at that point. And it was shown in Art. 226 that, for a portion of a beam bent into the form of a circular arc,

$$M = \frac{EI}{R}$$

Hence, for any beam,

$$M = \pm EI\frac{\frac{d^2y}{dx^2}}{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and for a beam in which $\frac{dy}{dx}$ is everywhere small,

$$M = \pm EI\frac{d^2y}{dx^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The ambiguity in sign is removed by deciding which we shall call a positive bending moment. The rule in Art. 227 is that the bending moment is positive when the beam is convex upwards. Since y is measured downwards, a figure will show that if a beam is convex upwards $\frac{dy}{dx}$ increases with x; that is, $\frac{d^2y}{dx^2}$ is positive. Thus, $\frac{d^2y}{dx^2}$ is positive when M is positive. Hence we must take the positive sign in (4). Thus—

$$M = EI\frac{d^2y}{dx^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The useful equations in this subject are (1) and (2) of Art. 229, and (5) above. We will write the first two of these equations here again for convenience—

$$\frac{dF}{dx} = -w \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$\frac{dM}{dx} = -F \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Differentiating both sides of (5), we get

$$EI\frac{d^3y}{dx^3} = \frac{dM}{dx}$$

$$= -F \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Differentiating again, $$\mathrm{EI}\frac{d^4y}{dx^4} = -\frac{d\mathrm{F}}{dx}$$
$$= w \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Equations (8) and (9) are only correct if I is constant along the beam, as in the case of a beam with uniform cross-section. If I is not constant, we get, instead of (8) and (9),

$$\mathrm{E}\frac{d}{dx}\left(\mathrm{I}\frac{d^2y}{dx^2}\right) = -\mathrm{F} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8a)$$

$$\mathrm{E}\frac{d^2}{dx^2}\left(\mathrm{I}\frac{d^2y}{dx^2}\right) = w \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9a)$$

For the present we shall deal only with uniform beams, so that we can take I to be constant.

When w is given, equations (5), (7), and (9), enable us to express y in terms of x and constants of integration. These constants have to be determined by the conditions to which the beam is subjected at the ends. But if there is a finite load or any kind of support at a point of the beam other than the ends, the constants will be different on opposite sides of that point.

There are two conditions at each end of a beam, giving four conditions in all, and since there will be four constants in the integral of (9), these four conditions are just sufficient to determine the constants. The conditions are as follows :—

At a supported end—

(1) y is known ; (2) M is zero, *i.e.* $\frac{d^2y}{dx^2}$ is zero.

At a fixed end—

(1) y is known; (2) $\frac{dy}{dx}$ is known, and is usually zero.

At a free end—

(1) F is equal to the load at the end; (2) M is zero.

231. Uniform Beam fixed at Both Ends under a Uniform Load.— A uniform beam is built into a wall at each end, the two ends being at the same level, and $\frac{dy}{dx}$ is zero at each end. The load, w, per unit length is constant. This load may be the weight of the beam itself. To find F, M, and y, in terms of x.

Let the origin be taken at the left-hand end of the beam. The differential equation to be solved is

$$\mathrm{EI}\frac{d^4y}{dx^4} = w \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Integrating four times in succession, we get

$$\mathrm{EI}\frac{d^3y}{dx^3} = wx + \mathrm{A} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\mathrm{EI}\frac{d^2y}{dx^2} = \tfrac{1}{2}wx^2 + \mathrm{A}x + \mathrm{B} \quad \ldots \ldots \quad (3)$$

$$\mathrm{EI}\frac{dy}{dx} = \tfrac{1}{6}wx^3 + \tfrac{1}{2}\mathrm{A}x^2 + \mathrm{B}x + \mathrm{C} \quad \ldots \ldots \quad (4)$$

$$\mathrm{EI}y = \tfrac{1}{24}wx^4 + \tfrac{1}{6}\mathrm{A}x^3 + \tfrac{1}{2}\mathrm{B}x^2 + \mathrm{C}x + \mathrm{D} \quad (5)$$

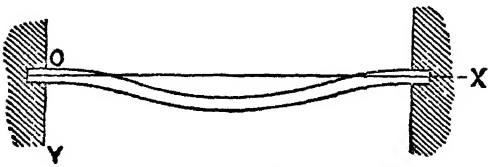


FIG. 122.

If the length of the beam is l, the end-conditions are

$$\left.\begin{array}{r} y = 0 \\ \text{and} \quad \dfrac{dy}{dx} = 0 \end{array}\right\} \begin{array}{l} \text{when } x = 0, \text{ and} \\ \text{when } x = l \end{array} \quad \ldots \ldots \quad (6)$$

The conditions when $x = 0$ give

$$\left.\begin{array}{r} \mathrm{D} = 0 \\ \mathrm{C} = 0 \end{array}\right\} \quad \ldots \ldots \quad (7)$$

The other two conditions give

$$\tfrac{1}{24}wl^4 + \tfrac{1}{6}\mathrm{A}l^3 + \tfrac{1}{2}\mathrm{B}l^2 = 0 \quad \ldots \ldots \quad (8)$$

$$\tfrac{1}{6}wl^3 + \tfrac{1}{2}\mathrm{A}l^2 + \mathrm{B}l = 0 \quad \ldots \ldots \quad (9)$$

From (8) and (9)—

$$\mathrm{A} = -\tfrac{1}{2}wl; \ \mathrm{B} = \tfrac{1}{12}wl^2 \quad \ldots \ldots \quad (10)$$

Substituting, in equations (2) to (4), the values found for the constants, we get

$$\left.\begin{array}{l} \mathrm{F} = -\mathrm{EI}\dfrac{d^3y}{dx^3} = -\tfrac{1}{2}w(2x - l) \\ \mathrm{M} = \mathrm{EI}\dfrac{d^2y}{dx^2} = \tfrac{1}{12}w(6x^2 - 6lx + l^2) \\ \mathrm{EI}y = \tfrac{1}{24}wx^2(l - x)^2 \end{array}\right\} \quad \ldots \quad (11)$$

The maximum deflection is clearly at the middle of the beam where $x = \frac{l}{2}$, and its value is

$$\frac{1}{\mathrm{EI}} \cdot \frac{1}{384}wl^4 \quad \ldots \ldots \quad (12)$$

For a uniform beam fixed horizontally at the end where $x = 0$ and free at the other, under a uniform load w per unit length, the same method will give

$$EIy = \frac{1}{24} wx^2 (6l^2 - 4lx + x^2)$$

And for a uniform beam supported on pegs at the ends, where $x = 0$ and $x = l$, the same method gives

$$EIy = \frac{1}{24} wx (x^3 - 2l x^2 + l^3).$$

The student is recommended to work these out.

232. Method of dealing with a Beam with a Concentrated Load.—As we have pointed out before, the shearing force is discontinuous at a concentrated load. Consequently the expressions for y on opposite sides of the load are different. We have therefore to find two sets of constants, one set to apply to each portion of the beam. There will be eight constants to be found, and we shall want eight equations to find them from. Four of these equations are given by the end-conditions of the beam. The other four are given by the relations between y and its differential coefficients infinitely near the load and on opposite sides of it. We will indicate what these relations are.

If a known load is applied at any point of a beam,

(1) the difference of the shearing force on the two sides of the load is equal to the load itself;
(2) the bending moment remains unaltered on passing the load.
(3) *and* (4) y and $\frac{dy}{dx}$ remain unaltered on passing the load.

If, instead of a known load being applied, a support is placed under the beam at a known height, then the conditions are—

(1) *and* (2) y is known at the point of support for both portions of the beam;
(3) *and* (4) the bending moment and $\frac{dy}{dx}$ remain unaltered on passing the load.

An example will illustrate the method.

Both ends of a uniform beam are fixed horizontally at the same height. If the length of the beam is l, *and a single load* W *is concentrated at a point distant* a *from one end, to find the deflection, neglecting the weight of the beam itself.*

We shall indicate quantities in the portions of the beam of lengths a and $(l - a)$ by suffixes 1 and 2 respectively, and we will suppose the former portion is the left-hand one.

Between the load and either end $w = 0$. Hence in each portion of the beam

$$EI\frac{d^4y}{dx^4} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Solving this for both portions of the beam, we get

$$EIy_1 = \tfrac{1}{6}A_1x^3 + \tfrac{1}{2}B_1x^2 + C_1x + D_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$
$$EIy_2 = \tfrac{1}{6}A_2x^3 + \tfrac{1}{2}B_2x^2 + C_2x + D_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Now we have to find the eight constants in these two equations from our eight conditions. The conditions at the load are

$$\left.\begin{aligned} F_1 - F_2 &= W \\ M_1 - M_2 &= 0 \\ \frac{dy_1}{dx} - \frac{dy_2}{dx} &= 0 \\ y_1 - y_2 &= 0 \end{aligned}\right\} \text{ when } x = a \quad . \quad . \quad (4)$$

After finding F_1, F_2, etc., from equations (2) and (3), and substituting in these conditions, we get

$$A_1 - A_2 = -W \quad . \quad . \quad . \quad (5)$$

$$(A_1 - A_2)a + (B_1 - B_2) = 0 \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$\tfrac{1}{2}(A_1 - A_2)a^2 + (B_1 - B_2)a + (C_1 - C_2) = 0 \quad (7)$$

$$\tfrac{1}{6}(A_1 - A_2)a^3 + \tfrac{1}{2}(B_1 - B_2)a^2 + (C_1 - C_2)a + (D_1 - D_2) = 0 \quad (8)$$

These four equations give

$$B_1 - B_2 = aW \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$C_1 - C_2 = -\tfrac{1}{2}a^2W \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$D_1 - D_2 = \tfrac{1}{6}a^3W \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Now the end-conditions are

$$\left.\begin{aligned} y_1 &= 0 \\ \frac{dy_1}{dx} &= 0 \end{aligned}\right\} \text{ when } x = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

$$\left.\begin{aligned} y_2 &= 0 \\ \frac{dy_2}{dx} &= 0 \end{aligned}\right\} \text{ when } x = l \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

The four conditions give

$$C_1 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

$$D_1 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

$$\tfrac{1}{6}A_2l^3 + \tfrac{1}{2}B_2l^2 + C_2l + D_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$$\tfrac{1}{2}A_2l^2 + B_2l + C_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Now equations (10), (11), (14), (15) give C_1, C_2, D_1, D_2, immediately. Then A_2 and B_2 are found from (16) and (17) after substituting for C_2 and D_2. Thus—

$$\left.\begin{aligned} A_2 &= \frac{a^2W}{l^3}(3l - 2a) \\ B_2 &= -\frac{a^2W}{l^2}(2l - a) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

These two equations combined with (5) and (9) give

$$\left.\begin{aligned} A_1 &= -\frac{W}{l^3}(l - a)^2(l + 2a) \\ B_1 &= \frac{aW}{l^2}(l - a)^2 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (19)$$

On substituting the values found for the constants, equations (2) and (3) become

$$\mathrm{EI}y_1 = -\frac{\mathrm{W}x^2}{6l^3}(l-a)^2\{(l+2a)x - 3al\} \quad . \quad . \quad . \quad . \quad . \quad (20)$$

$$\mathrm{EI}y_2 = +\frac{\mathrm{W}a^2}{6l^3}\{(3l-2a)x^3 - 3l(2l-a)x^2 + 3l^3x - l^3a\}$$

$$= +\frac{\mathrm{W}a^2}{6l^3}(l-x)^2\{(3l-2a)x - al\}. \quad . \quad . \quad . \quad . \quad . \quad (21)$$

233. Deflection due to Several Systems of Loads.—Suppose the end-conditions of a beam are that y or some of its differential coefficients are zero; and suppose we use these conditions to find the deflections due to two separate systems of loads. Then the deflection due to the sum of the two systems of loads is the sum of the deflections due to each separately.

Let y_1 and y_2 be the deflections due to two systems of loads w_1 and w_2 per unit length, and let $y = y_1 + y_2$. We shall show that, under the conditions stated, y is the deflection due to both loads.

For, since $$y = y_1 + y_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

therefore $$\mathrm{EI}\frac{d^4y}{dx^4} = \mathrm{EI}\frac{d^4y_1}{dx^4} + \mathrm{EI}\frac{d^4y_2}{dx^4}$$

$$= w_1 + w_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Thus y satisfies the differential equation for the deflection with a load $(w_1 + w_2)$ per unit length.

Again, the constants of integration which appear in y_1 and y_2 are determined by making these quantities or their differential coefficients equal to zero at the ends. But since

$$\frac{d^ny}{dx^n} = \frac{d^ny_1}{dx^n} + \frac{d^ny_2}{dx^n}$$

it follows that the term on the left is zero when both terms on the right are zero. That is, y and its differential coefficients satisfy the same boundary conditions as y_1 and y_2 and their differential coefficients. Thus y, which equals $y_1 + y_2$, satisfies all the conditions for the deflection of the beam under the load $(w_1 + w_2)$. Therefore $(y_1 + y_2)$ must be the deflection with this load.

The preceding argument will hold when one or both of the systems of loads is a set of isolated loads. For suppose one of the systems includes a load W at a given point. Then the solution for this system gives an increase of the shear by an amount W on passing this load, and the solution for the other system gives no sudden alteration of the shear. Thus the sum of the two solutions gives an increase of the shear by W, which is as it should be for the sum of the loads. The rest of the conditions are just the same as before.

As an example of the preceding, if we want the deflection due to a

continuous load w per unit length and a finite load W at one point of a beam built into a wall at both ends, we have only to add the deflections obtained in Arts. 231 and 232, where the two loads are treated separately.

But if the conditions of the end of the beam are that y or some of its differential coefficients are given quantities, not zero, then we must not use the same boundary conditions for both systems of loads. If, for example, $\frac{d^3y}{dx^3} = K$, a known quantity, at one end of the beam, we must use this condition for one system of loads, and make $\frac{d^3y}{dx^3} = 0$ at this end for the other system.

Or again, suppose there is a support at one point of a beam so that y is known there. We can assume that the support applies a load (positive or negative) at that point. We may call this unknown load W, and find the deflection due to W and the other given loads. Then equating the given deflection at the given point to the calculated deflection, we can find W, and thence the deflection at any other point.

EXAMPLE.—*A uniform beam of length* l *is fixed horizontally at both ends and is supported by a prop at a distance* a *from one end, so that the deflection is zero at that point. To find the pressure on the support and the deflection at any point of the beam.*

Let R be the upward pressure of the prop. Then the deflection due to R alone is given by putting $-R$ for W in equations (20) and (21) of Art. 232. Also the deflection due to the weight of the beam alone is given by the last of equations (11) in Art. 231. Hence the deflection due to both is y, given by

$$EIy = \tfrac{1}{24}wx^2(l-x)^2 + \frac{Rx^2}{6l^3}(l-a)^2\{(l+2a)x - 3al\} \quad . \quad (1)$$

in the portion of length a; and

$$EIy = \tfrac{1}{24}wx^2(l-x)^2 - \frac{Ra^2}{6l^3}(l-x)^2\{(3l-2a)x - al\} \quad . \quad (2)$$

in the other portion.

The two values of y in (1) and (2) are equal when $x = a$, and this common value must be zero. Hence, putting $y = 0$ and $x = a$ in (1), we get

$$R = \tfrac{1}{8}w\frac{l^3}{a(l-a)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

On substituting this value for R in (1) and (2), we get the deflection at any point of the beam.

234. Beams with Variable Sections.—It was pointed out in Art. 230 that if I is variable we must use equation ($9a$) instead of equation (9) of that Article. Also the shear is given by ($8a$), and the equations (6) and (7), being independent of I, are true for all beams. We will work an example to show the method to be used.

The thick end of a wedge-shaped beam is built into a wall, the edge of the wedge being vertical and unsupported. To find F, M, and y, when the only load is the weight of the beam itself. Let l be the length of the beam, d the depth, and b the width of the thick end. Let ρ be the density of the material. Taking the origin at the fixed end, the width at x is $\frac{l-x}{l}b$. Thus the weight of a length dx is $\rho\frac{l-x}{l}b\,.\,ddx$. Hence the weight per unit length at x is

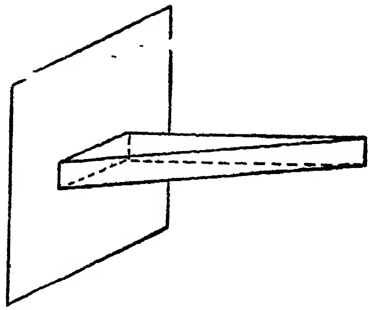


FIG. 123.

$$w = \rho\frac{l-x}{l}bd \quad \ldots \ldots \quad (1)$$

Also
$$I = \frac{1}{12}\cdot\frac{l-x}{l}bd^3 \quad \ldots \ldots \quad (2)$$

Now the fundamental equation is

$$E\frac{d^2}{dx^2}\left(I\frac{d^2y}{dx^2}\right) = w = \rho\frac{l-x}{l}bd \quad \ldots \ldots \quad (3)$$

Integrating this, $-F = E\frac{d}{dx}\left(I\frac{d^2y}{dx^2}\right) = -\frac{1}{2}\cdot\frac{\rho bd}{l}(l-x)^2 + A \quad . \quad . \quad (4)$

And again $\quad M = EI\frac{d^2y}{dx^2} = +\frac{1}{6}\cdot\frac{\rho bd}{l}\cdot(l-x)^3 + Ax + B \quad . \quad (5)$

Now at the free end, where $x = l$, we have the conditions

$$F = 0,\ M = 0 \quad \ldots \ldots \quad (6)$$

Whence
$$A = 0,\ B = 0 \quad \ldots \ldots \quad (7)$$

Now putting A and B zero in (5) and substituting for I from (2) we get

$$E\frac{1}{12}\cdot\frac{l-x}{l}bd^3\cdot\frac{d^2y}{dx^2} = \frac{1}{6}\cdot\frac{\rho bd}{l}(l-x)^3 \quad \ldots \ldots \quad (8)$$

Thus
$$\frac{d^2y}{dx^2} = \frac{2\rho}{Ed^2}(l-x)^2 = \frac{2\rho}{Ed^2}(l^2 - 2lx + x^2) \quad (9)$$

Integrating twice and introducing the conditions that $y = 0$ and $\frac{dy}{dx} = 0$ when $x = 0$, we get

$$y = \frac{2\rho}{Ed^2}\left(\tfrac{1}{2}l^2x^2 - \tfrac{1}{3}lx^3 + \tfrac{1}{12}x^4\right)$$
$$= \frac{\rho x^2}{6Ed^2}(6l^2 - 4lx + x^2) \quad \ldots \ldots \quad (10)$$

Thus y is found. Also F and M are given by equations (4) and (5) on putting A and B each equal to zero.

In a case like this it is easy to get shearing force and bending moment without having recourse to the equation for the deflection. By equation (1), Art. 227

$$F = \text{weight of the portion of the beam to the right of } x$$
$$= \rho \tfrac{1}{2} b \frac{l-x}{l}(l-x)d$$
$$= \frac{\rho db}{2l}(l-x)^2$$

and $$M = \text{the moment of this weight about the point } x$$
$$= \tfrac{1}{3}(l-x) \, . \, F$$
$$= \frac{\rho bd}{6l}(l-x)^3$$

and these are the same values as before.

235. Strength of Bent Beams.—When breaking occurs at any section of a bent beam, the first point to give way is either the one on the stretched side of the neutral axis furthest from that axis, or the one furthest away on the compressed side. Whether the stretched or the compressed side will break first depends on the shape of the section and the relative magnitudes of the tensile and compressive breaking stresses. Now suppose P is the maximum safe tensile stress and h the greatest distance of any point of the section from the neutral axis on the stretched side. It was proved in Art. 226 that the strain at distance h from the axis is $\frac{h}{R}$. Hence the stress there is $E\frac{h}{R}$. Now if this is the maximum safe stress, we have

$$P = E\frac{h}{R} \quad \dots\dots\dots \quad (1)$$

But if M is the bending moment at the section

$$M = E\frac{I}{R} \quad \dots\dots \quad (2)$$

Hence $$M = \frac{IP}{h} \quad \dots\dots\dots\dots \quad (3)$$

Again, if Q is the maximum safe compressive stress, and k the greatest distance of any point of the section from the neutral axis on the compressed side, the bending moment M_1 corresponding to the stress Q at the extremity is

$$M_1 = \frac{IQ}{k} \quad \dots\dots\dots \quad (4)$$

The maximum safe bending moment at the section is therefore the smaller of the two quantities M and M_1 given by (3) and (4).

If the section of the beam is rectangular of width b and depth d, and there is no resultant tension across the section, then

$$I = \tfrac{1}{12}bd^3, \qquad h = \frac{d}{2}, \qquad k = \frac{d}{2} \quad . \quad . \quad . \quad . \quad (5)$$

Hence $$M = \tfrac{1}{6}bd^2P, \qquad M_1 = \tfrac{1}{6}bd^2Q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Thus the strength of the beam to resist breaking by bending is proportional to the breadth and the square of the depth.

We will compare the strengths of two beams with similar sections of different size. Let b be the maximum width. Then it is clear that

$$I = Cbh^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where C is some number and the same for all similar sections. Hence

$$M = Cbh^2P \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Now h is proportional to the depth of the section. It follows, therefore, that the safe bending moments for similar sections varies as the breadth and the square of the depth. If d is the depth, we may write (8) thus—

$$M = Abd^2P \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The special case of a rectangular section has already been worked out separately. For this section $A = \frac{1}{6}$ as in equation (6). For a circular section, or for an ellipse with a principal axis horizontal, $A = \dfrac{\pi}{32}$.

236. Comparison of Deflections due to Shearing and Bending.—It must not be forgotten that a beam with transverse loads has a deflection due to shear strain as well as the one due to bending. The true deflection is the sum of these two. In the case of beams whose lengths are several times their depths, the deflection due to shear strain is small enough to be ignored in comparison with the other deflection. But for short beams the shear deflection may be as important as the bending deflection. To illustrate this let us take, as an example, a uniform beam with any shape of section, having a load W at one end and built into a wall at the other. We shall neglect the weight of the beam itself.

Let l be the length of the beam, y_1 and y_2 the deflections at the end due to bending and shearing respectively. Let A be the area of the section, Ak^2 its moment of inertia about the neutral axis. Then if x is measured from the fixed end, the deflection due to bending is given by

$$EIy = W\left(\tfrac{1}{2}lx^2 - \tfrac{1}{6}x^3\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

At the free end, where $x = l$ and $y = y_1$, we get

$$y_1 = \frac{W}{EI} \cdot \frac{1}{3}l^3 = \frac{W}{EAk^2} \cdot \frac{1}{3}l^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Also, for the shear strain, n being the modulus of rigidity,

$$\frac{W}{A} = n\frac{y_2}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From (2) and (3) we get
$$\frac{y_2}{y_1} = \frac{3Ek^2}{nl^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

For cast iron $E = 18 \times 10^6$ lbs. per square inch

$n = 6 \times 10^6$ lbs. per square inch

With these values
$$\frac{y_2}{y_1} = \frac{9k^2}{l^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

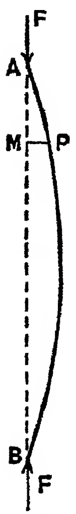
If the beam has a rectangular section of depth d, then $k^2 = \frac{1}{12}d^2$. Therefore

$$\frac{y_2}{y_1} = \frac{3d^2}{4l^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Thus, if $l = d$, the two deflections are nearly equally important.

237. Theory of Struts.—A strut is a long thin naturally straight rod subject to a thrust. In the following theory it is assumed that we are dealing with uniform rods whose lengths are considerably greater than their widths. If the length in any case is greater than forty times the width, the results obtained will be fairly accurate.

Suppose two equal and opposite forces F act at the ends A and B of a strut, these forces being in equilibrium. Let y be the deflection from the straight line AB of any point P at distance x from A. Thus in the figure AM $= x$, MP $= y$. Now exactly as in beams with lateral loads



$$EI\frac{d^2y}{dx^2} = \text{bending moment at P}$$
$$= \text{moment of external forces on AP}$$
$$= -Fy \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The reason for taking $-Fy$ and not $+Fy$ is because it is obvious from the figure that $\frac{d^2y}{dx^2}$ is negative, since $\frac{dy}{dx}$ is decreasing as x increases, and we cannot have a negative quantity equal to a positive quantity.

Equation (1) may be written

$$\frac{d^2y}{dx^2} = -\frac{F}{EI}y$$
$$= -c^2y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

FIG. 124. where c is written for $\sqrt{\frac{F}{EI}}$ for the sake of brevity.

Equation (2) is one which occurs very often in dynamics. In all questions of small oscillations of a body about a position of stable

equilibrium we have to solve an equation of this type. The solution is obtained in Art. 292, in connection with small oscillations of a particle. We give here an alternative method of solution, which the student had better pass over if he knows nothing of imaginary quantities.

Put

$$y = ue^{icx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where i means $\sqrt{-1}$, and therefore $i^2 = -1$.

Differentiating both sides of (3) twice, we get

$$\frac{d^2y}{dx^2} = e^{icx} \cdot \frac{d^2u}{dx^2} + 2ice^{icx} \cdot \frac{du}{dx} - c^2e^{icx}u \quad . \quad . \quad . \quad (4)$$

that is, since the last term is $-c^2y$,

$$\frac{d^2y}{dx^2} + c^2y = e^{icx}\left(\frac{d^2u}{dx^2} + 2ic\frac{du}{dx}\right) . \quad . \quad . \quad . \quad . \quad (5)$$

But by (2) the left-hand side of this equation is zero. Therefore

$$\frac{d^2u}{dx^2} + 2ic\frac{du}{dx} = 0 \, . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Integrating this, $\qquad \dfrac{du}{dx} + 2icu = \text{a constant}$

$$= 2icA \text{ say} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Hence $\qquad 2ic\dfrac{dx}{du} = -\dfrac{1}{u - A} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$

and therefore $\qquad 2icx = -\log_e (u - A) + \log_e B \quad . \quad . \quad (9)$

where, for convenience, $\log_e B$ is written for the constant of integration.

Equation (9) can be transformed into

$$\frac{B}{u - A} = e^{2icx}$$

or $\qquad u = A + Be^{-2icx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$

whence $\qquad y = Ae^{icx} + Be^{-icx}$

$$= A(\cos cx + i \sin cx) + B(\cos cx - i \sin cx)$$

$$= (A + B) \cos cx + i(A - B) \sin cx \, . \quad . \quad . \quad . \quad (11)$$

And, since A and B are any constants, and may be such complex quantities as will make $(A + B)$ and $i\,(A - B)$ both real, we may write

$$y = C \cos cx + D \sin cx \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

This is the general solution of equation (2). We have now to make it fit our problem.

Let $l =$ the length of the rod. Then we know that $y = 0$ when $x = 0$, and $y = 0$ when $x = l$. The first of these conditions gives

$$0 = C \, . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and then the second gives

$$0 = D \sin cl \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Now (14) can be satisfied either by taking $D = 0$, or $\sin cl = 0$. If $D = 0$ there is no deflection of the rod anywhere, and the rod is therefore straight. But taking

$$\sin cl = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

we get, as possible solutions,

$$cl = \pi,\ 2\pi,\ 3\pi, \text{ etc.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

This gives us c, and therefore F in terms of l. Thus, substituting for c and squaring,

$$\frac{F}{EI} = \frac{\pi^2}{l^2}, \qquad \frac{4\pi^2}{l^2}, \qquad \frac{9\pi^2}{l^2}, \text{ etc.} \quad . \quad . \quad . \quad (17)$$

Also substituting for c in (12),

$$y = D \sin \frac{\pi x}{l} \quad \text{or} \quad D \sin \frac{2\pi x}{l}, \text{ etc.} \quad . \quad . \quad . \quad (18)$$

If the rod is bent and in equilibrium, F must have one of the values given by (17). A force less than the smallest of these forces is not capable of bending the rod. The smallest force, namely $EI\frac{\pi^2}{l}$, bends the rod so that y is zero only at the ends. The second force $EI\frac{4\pi^2}{l}$ bends the rod so that y is zero at the middle as well as at the ends, as is shown by taking the second result in (18). In practice the rod could not retain this second form unless there were some constraint at the middle preventing that point from leaving the straight line AB. If no constraints are applied any force greater than the smallest will bend the rod; but as our present theory applies only to infinitesimal deflections it will not tell us precisely what force will break the rod.

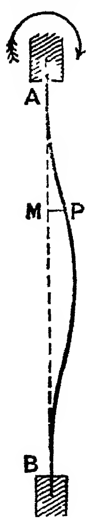


FIG. 125.

Perhaps the best way to realise the meaning of the result is the following. If the rod is placed with its ends against two points whose distance apart is less than the length of the rod, or if the ends are tied together by a string, the thrust exerted by each end of the rod against the constraining bodies, or the tension in the string, is equal to $EI\frac{\pi^2}{l^2}$ whether the distance between the ends is only just less or considerably less than the length of the rod.

238. Strut built in at the Ends.—If the ends of a strut are fixed so that $\frac{dy}{dx}$ is zero there, we shall have to modify equation (2) a little to adapt it to this case. At the fixed end the bending moment is not zero. Let N be the couple

applied at the end A. Then the moment about P of the forces on AP is

$$-Fy + N$$

Hence, equating this to the bending moment at P,

$$EI\frac{d^2y}{dx^2} = -Fy + N$$

$$= -F\left(y - \frac{N}{F}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now, putting $z = y - \frac{N}{F}$, we get

$$\frac{d^2z}{dx^2} = \frac{d^2y}{dx^2}$$

and therefore
$$EI\frac{d^2z}{dx^2} = -Fz \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

of which we know the solution to be

$$z = C\cos cx + D\sin cx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

whence
$$y = \frac{N}{F} + C\cos cx + D\sin cx \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The conditions at the ends are now

$$\frac{dy}{dx} = 0 \left\{ \begin{array}{l} \text{when } x = 0 \\ \text{and } x = l \end{array} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$y = 0 \left\{ \begin{array}{l} \text{when } x = 0 \\ \text{and } x = l \end{array} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The first two conditions give

$$D = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$cC\sin cl = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Then the two remaining conditions give

$$\frac{N}{F} + C = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\frac{N}{F} + C\cos cl = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

From the last two equations we get

$$C(1 - \cos cl) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Now if the rod is bent at all C is not zero, and therefore equations (8) and (11) tell us that

and
$$\left. \begin{array}{r} \sin cl = 0 \\ \cos cl = 1 \end{array} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Since both these equations must be true simultaneously, we get

$$cl = 2\pi, \qquad 4\pi, \qquad 6\pi, \text{ etc.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and therefore
$$\frac{F}{EI} = \frac{4\pi^2}{l^2}, \qquad \frac{16\pi^2}{l^2}, \qquad \frac{36\pi^2}{l^2}, \text{ etc.} \quad . \quad . \quad . \quad . \quad (14)$$

The equations do not tell us what N is. We only know that $\frac{N}{F} = -C$, but C is itself unknown. The final value for y may therefore be written

$$y = -C\left(1 - \cos\frac{2\pi x}{l}\right), \text{ or } = -C\left(1 - \cos\frac{4\pi x}{l}\right), \text{ etc.} \quad . \quad . \quad (15)$$

In this case the smallest force that will bend the rod is four times as great as the smallest force required to bend the same rod when the ends are not clamped.

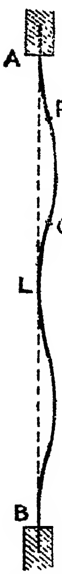


FIG. 126.

In all cases the form taken by a strut is a portion of a sine curve. When the ends are not fixed these ends are points of inflection on the sine curve, and thus the rod may have the form of any whole number of half wave-lengths of a sine curve. But when the ends are fixed these ends are both crests of a sine curve, (or both troughs). Thus the rod takes the form of a whole number of wave-lengths. If the bodies to which the ends of the rod are fixed are at a given distance apart, that distance being less than the length of the rod, constraints will be needed to make the rod take any form except that of a whole wave-length. Thus, in order to make it take the form of two wave-lengths, the middle point L (Fig. 126) must be fixed in the straight line AB. Without a constraint this form gives a possible position of equilibrium, but it is unstable. The constraining body need apply no force at L except to balance such small external forces as would disturb the equilibrium.

At a point of inflexion of the rod there is no bending moment, because $\frac{d^2y}{dx^2}$ is zero. It is clear, then, that the action at one of these points is a single force F parallel to AB. Thus if P, Q, are two consecutive points of inflexion, the length PQ is exactly similar to the whole rod in the first case where the directions of the ends were not fixed. Since the length of PQ is $\frac{1}{4}l$, we can get F from the first example by putting $\frac{1}{4}l$ for l in the smallest value found for F in that example. This gives

$$\frac{F}{EI} = \frac{16\pi^2}{l^2}$$

which is the same as the second value of F for a rod with fixed ends, and this we know to be correct.

EXAMPLES ON CHAPTER X

1. A uniform indiarubber string has a length of 26 inches under a tension of $2\frac{1}{2}$ lbs, and a length of 20 inches under a tension of one lb. Calculate the amount of work done in stretching it from its natural length to a length of 30 inches, and draw a work diagram. *London B.Sc.*

[The work done is $2\frac{1}{24}$ foot-lbs. A work diagram is force plotted against the displacement of the particle to which it is applied, and is a straight line in this case.]

2. Prove that the potential energy stored up in a stretched string is half the product of the tension and the extension.

A uniform bar, 6 feet long, weighing 20 lbs., lies on perfectly rough ground. Evaluate the work done in raising one end from the ground to a height of 3 feet by means of a vertical string attached to that end,

(i) when the string is inelastic;

(ii) when the string is elastic, 2 feet long, and such that its length would be doubled by a pull of 30 lbs. weight.

London B.Sc.

[(i) 30 foot-lbs.; (ii) $33\frac{1}{3}$ foot-lbs.]

3. A string, whose natural length is na, has n equal weights attached to it at distances a, $2a$, $3a$, etc., from the highest point which is fixed. Find the total increase of length and the potential energy.

Manchester University B.Sc.

[If w denotes one of the weights, and if tension $= k \times$ (strain), the increase of length is $\frac{1}{2}n(n+1)\dfrac{wa}{k}$, and the potential energy is $\frac{1}{6}n(n+1)(2n+1)\dfrac{w^2a}{2k}$.]

4. An elastic string of length l has two equal weights w attached to it, one at the middle and one at an end of the string, and the whole lies on the ground. The string is picked up by the free end and is kept vertical until both weights are off the ground. If one of the weights would extend the whole string by nl, find the least work necessary to lift both weights off the ground.

$$\left[wl\left(\frac{1}{2}+\frac{7n}{4}\right)\right]$$

5. If there were three weights attached to the string referred to in the last question at distances $\frac{1}{3}l$, $\frac{2}{3}l$, and l from the free end, find the least amount of work necessary to lift all the weights by the free end of the string.

$$\left[wl\left(1+\frac{11n}{3}\right).\right]$$

6. A uniform elastic string of length $2l$ is suspended between two points at the same level and hangs in the form of the catenary $y = c\cosh\dfrac{x}{c}$, its length being only slightly increased by the tension in the string. Show that the potential energy of the elastic forces is $\dfrac{\rho^2 a}{E}(c^2l + \frac{1}{3}l^3)$, ρ being the density, and a the area, of the section of the string.

7. The ends of an elastic string are attached to two points in a vertical line at a greater distance apart than the natural length of the string. If a weight W is now attached to the middle of the string, show that, assuming the lower half does not become slack, it will displace that middle point one quarter as much as it would extend the whole string if the end to which it were attached were otherwise free.

8. An elastic string, of natural length l and density ρ, is attached to two fixed points in a vertical line so that its extension is b, which is greater than the weight of the string itself would produce. If E is Young's modulus for the material, and a the cross-section of the string, show that the whole displacement of a point P, whose unstretched distance from the top would be x, is

$$\frac{\rho}{2\mathrm{E}}x(l-x)+\frac{b}{l}x$$

Also find the tension at the lowest point and the potential energy of the stretched string.

$$\left[\text{Tension} = \frac{a}{2l}(2\mathrm{E}b-\rho l^2);\quad \text{Pot. E} = \frac{a}{24\mathrm{E}l}(12\mathrm{E}^2b^2+\rho^2l^4).\right]$$

9. One end of an elastic string, of length l and negligible weight, is fastened to a fixed point A at the same level as the top of a rough horizontal cylindrical rod and at distance b from it. The other end is passed over the rod, and then a weight w is attached to this end and allowed to reach its equilibrium position slowly. If the tension in the string is the product of nw and the strain and the coefficient of friction between the string and the rod is μ, find by how much the weight w stretches the whole string, the length in contact with the rod being negligible.

$$\left[\frac{l}{n}-\frac{b\left(e^{\frac{\mu\pi}{2}}-1\right)}{ne^{\frac{\mu\pi}{2}}+1}.\right]$$

10. A uniform extensible string, of length l in its natural state, rests on a rough inclined plane of inclination θ, with its upper end fixed to the plane; prove that its extension will lie between the limits $\frac{l^2}{2c}\cdot\frac{\sin(\theta\pm\epsilon)}{\cos\epsilon}$, where ϵ is the angle of friction, and c is the length of a portion of the string in its natural state whose weight is equal to the modulus of elasticity.

Dublin University Term exams.

[Show firstly that the tension is $\rho x(\sin\theta\pm\tan\epsilon\cos\theta)$ at distance x from the lower end in the two limiting positions, ρ being the weight per foot.]

11. Six uniform rods of the same material and diameter form the sides and diagonals of a rhombus ABCD with smooth hinges at the corners, and the lengths are such that when no forces act on the frame the rods are unstressed. The frame is suspended by the hinge A, and a weight W is attached to C. If AB = l, AC = $2a$, BD = $2b$, show that the tensions in these three rods are respectively

$$\frac{a^2l}{d}\mathrm{W},\quad \frac{l^3+2l^3}{d}\mathrm{W},\quad -2\frac{a^2b}{d}\mathrm{W}$$

where $d = l^3+2a^3+2b^3$.

12. Three rods, AP, BP, CP, of the same material and diameter and of lengths a, b, c, are hinged together at P, and the other ends are fastened by hinges to a vertical beam. When no forces act on the frame, the rods are

unstressed and CP is perpendicular to the wall. Also AB and BC are each equal to l, and remain unaltered by stresses in the rods. When a load W is applied at P, show that the tensions in the rods AP, BP, CP, are $a(2b^3 + c^3)S$, $b(a^3 - c^3)S$, $-c(a^3 + 2b^3)S$, where

$$S = \frac{W}{l(a^3+4b^3 + c^3)}$$

13. ABCD is a framework formed by four equal rods joined by smooth hinges, and another rod connects A and C. The hinges B and D are attached to fixed points on rigid supporting bodies, so that the frame spans a gap with AC vertical and all the other rods inclined at θ to AC. Assuming all the rods to be of the same material and thickness, and assuming that the distance BD remains constant, find the tensions in AB, AC, and BC, due to a weight W at the lowest point C.

$$\left[-\frac{W}{4\cos\theta}\left(\frac{1}{1+2\cos^3\theta}\right);\ \frac{W}{2}\left(\frac{1}{1+2\cos^3\theta}\right);\ \frac{W}{4\cos\theta}\left(\frac{1+4\cos^3\theta}{1+2\cos^3\theta}\right).\right]$$

14. If the conditions of the last problem were the same, except that a rod joins B and D, and the hinge B is free to slide along a smooth fixed horizontal support, show that the stresses will be obtained from those given as the answers to the last question by adding $2\sin^3\theta$ to the numerators and denominators of the fractions in the brackets.

15. Suppose a bicycle wheel has n spokes all in one plane and all at the same tension when no load is applied. If the rim of the wheel retains its shape when a load W is applied to the hub, show that the extension of a spoke inclined at θ to the highest spoke (assumed to be vertical) is $x\cos\theta$, where x denotes the extension of the highest spoke. Thence show that the additional tension in the highest spoke is $\frac{2W}{n}$, and in the spokes inclined at θ to this, $\frac{2W}{n}\cos\theta$.

$$\left[\textit{Note.}—1 + \cos^2\frac{2\pi}{n} + \cos^2\frac{4\pi}{n} + \cos^2\frac{6\pi}{n} + \ldots + \cos^2\frac{2(n-1)}{n}\pi = \tfrac{1}{2}n.\right]$$

16. Referring to the last question, work out separately the case of a wheel with only four spokes when they are all inclined at 45° to the vertical, and show that the tension in each of the upper spokes is increased by $\frac{\sqrt{2}}{4}W$, and that the tension in each of the lower spokes is decreased by the same amount.

17. Whether the highest spoke of the bicycle wheel in question 15 is vertical or not, prove that the tension in a spoke inclined at θ to the highest radius is $\frac{2W}{n}\cos\theta$, and that the hub descends the same distance in each case.

18. Find the couple necessary to twist one end of a bar, whose section is an equilateral triangle, through ϕ radians, the other end being fixed, given that the length is l, the side of a section b, and the modulus of rigidity n.

$$\left[\frac{\sqrt{3}}{80}\frac{b^4 n\phi}{l}.\right]$$

19. A solid cylindrical steel shaft has a length of 40 feet and diameter 4 inches, and it carries a pulley at its middle point, the ends being fixed. Taking n to be 118×10^5 lbs. per square inch, and the diameter of the pulley 18 inches, find what force must be applied to the rim of the pulley to turn it through one degree.

[About 4792 lbs.]

20. A spiral spring has 40 coils, very nearly horizontal, and the diameter of the coils is one inch; the diameter of the wire is $\frac{1}{6}$ of an inch. If it takes an axial pull of $14\frac{1}{2}$ lbs. to stretch it half an inch, what is the modulus of rigidity of the material?

[About 1202×10^4 lbs. per square inch.]

21. An iron bar, two inches in diameter, for which Young's modulus is 29×10^6 lbs. per square inch, is bent into the form of an arc of a circle of 400 feet diameter. Find the maximum stress at any point of a transverse section. [12,080 lbs. per square inch.]

Show further that if the stress be limited to 4 tons per square inch the diameter of the circle must not be less than 540 feet.

22. Define *neutral axis* and *radius of curvature*. If a homogeneous uniform bar with a circular section one inch in diameter is bent to a radius of 720 inches by a bending moment of 200 ft.-lbs., find the value of E.

Victoria University, Honours Engineering.

[352×10^5 lbs. per square inch.]

23. Find the bending moment at the middle of a beam due to a weight W applied at the middle, the ends being simply supported. If the beam has a breadth 6 inches and a depth 8 inches, find the maximum length so that the stress shall not exceed 5000 lbs. per square inch, the weight W being three tons. [15·87 feet.]

24. If ρ is the density of the material of a beam of rectangular section with length l, breadth b, depth d, simply supported at its ends, find the maximum bending moment due to its own weight. [$\frac{1}{8}\rho bdl^2$.]

If $\rho = \frac{1}{3}$ lb. per cubic inch, $d = 12$ inches, find the maximum span for a maximum safe stress of 2000 lbs. per square inch. [25·8 feet.]

25. A beam 20 feet long rests symmetrically on supports 14 feet apart. It is loaded with

(*a*) an evenly distributed load of half a ton per foot run over its whole length;

(*b*) 2 tons at one end.

Sketch a bending-moment diagram and calculate the bending moments over the supports and at the middle of the beam.

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[Bending moments at the support near the 2 tons, at the middle, and at the other support are $\frac{33}{4}$, -7, $\frac{9}{4}$, foot-tons. The vertex of the parabola representing the bending moment between the supports is at $10\frac{6}{7}$ feet from the 2-ton load, and at this point $M = -7\frac{9}{49}$ foot-tons.]

26. A beam rests on supports 15 feet apart and overhangs 7 feet at each end. It is loaded with an evenly distributed load of half a ton per foot run over the central span, 2 tons at one free end, and 3 tons at the other. Find the bending moment at the middle of the beam. Sketch shear and bending moment curves. *Victoria University, B.Sc. Engineering.*

[At the middle $M = 3\frac{7}{16}$ foot-tons. At $13\frac{17}{30}$ feet from the two-ton load the shear is zero, and the value of the bending moment, namely 3·220 foot-tons, is less than at any other point between the supports.]

27. A beam of length l, width b, depth d, is held by one end in a horizontal position so as to be bent by its own weight: show that the curvature at a distance x from the fixed end is $\dfrac{6w(l-x)^2}{Ebd^3}$. Find also the deflection at the other end of the beam.

$$\left[\frac{3wl^4}{2Ebd^3}.\right]$$

28. A horizontal beam AB, of uniform cross-section, is held firmly at the end A, but is bent slightly by its own weight: find the droop at B.

$$\left[\frac{Wl^3}{8EI}.\right]$$

If the end B were partly supported, so that the droop were one-half of what it would be without support, find the pressure on the support.

$$\left[\frac{3W}{16}.\right]$$

29. A beam of length l and weight W is fixed horizontally at one end and supported at the other so that the two ends are in the same horizontal line. Show that, at the supported end, the beam makes an angle

$$\tan^{-1}\frac{Wl^2}{48EI}$$

with the horizontal. *London Inter. Sci., Honours.*

30. A rod of uniform cross-section is supported horizontally on three supports, one at each end and one at the middle, so that there is no droop at the middle. Show that the greatest droops are very nearly at one-fifth of the length from each end.

31. If a uniform beam is supported at the same level at the ends and centre, show that there is no bending moment at three-eighths of the length of the beam from either end.

32. A beam is at first wholly supported on trestles at its ends, which are in a horizontal line; the middle of the beam is gradually pushed up by a trestle till the pressures on the end trestles are relieved. Show that the middle trestle has been raised $\frac{2}{5}$ of the droop at the middle in the first position, and determine the pressures at the instant when the trestles were in a straight line.

$$\left[\frac{3W}{16};\ \frac{5W}{8};\ \frac{3W}{16}.\right]$$

33. A uniformly loaded beam is supported at its ends by supports at the same level, and it is propped at the middle. If the centre prop is at the same level as the supports, find the points of zero bending moment. If now the centre prop be raised a distance equal to a quarter of the deflection of the centre when the prop is wholly removed, prove that it bears a pressure equal to $\frac{15}{32}$ of the whole load.

34. A beam, the moment of inertia of whose cross-section is I, is exposed a bending moment M. Show that the stress in any layer can be put in form $\frac{My}{I}$, where y is the distance of the layer from a certain plane.

A rectangular beam of breadth b, depth h, and length l, is built into a wall, the outer end being free; it carries a load, P, which is so distributed that the intensity of the load at any point varies as the distance from the free end: prove that the maximum stress is $\frac{2Pl}{bh^2}$.

Manchester University, Honours Engineering.

35. Three spans are bridged by two equal girders of uniform section, hinged together at the centre of the middle span. The girders rest freely on the four supports, which are all at the same level. Find the ratio of the

length of an outer to the middle span so that the neutral axis may be horizontal over the intermediate supports.

Manchester University, Honours Engineering.

[The spans are equal.]

36. A rectangular beam is symmetrically supported in a horizontal position upon two supports, the distance between which is $\frac{1}{\sqrt{3}}$ of the length of the beam. Determine the form assumed by the beam.

[Between the supports, at distance x from the middle,

$$y = \frac{12\rho}{Ed^2}\left(\frac{1}{24}x^4 - \frac{2\sqrt{3}-3}{48}l^2x^2 + \frac{12\sqrt{3}-19}{3456}l^4\right)$$

ρ being the density of the material, d the depth of the beam.]

37. A uniform beam of 15 feet span is fixed (not merely supported) at the ends and has two concentrated loads of one ton each at 5 feet from each end; it has also a uniformly distributed load of half a ton per foot run. Draw the bending moment and shearing force diagrams, and find the points of contraflexure [*i.e.* points of inflection where M is zero].

Victoria University Honours, Engineering.

[At x feet from one end, where $x < 5$,

$$M = \tfrac{1}{4}x^2 - \tfrac{19}{4}x + \tfrac{305}{24} \text{ foot-tons}; \quad M = 0 \text{ where } x = 3{\cdot}22 \text{ feet.}]$$

38 (*a*). If a horizontal beam of uniform section be loaded in any manner, and if M be the bending moment at distance x from any point on the beam with a given set of conditions at the end, show that, if the loads remain the same and the end conditions are altered in any way, the new bending moment will be $(M + Cx + B)$, where C and B are constants.

38 (*b*). If M is the bending moment of a uniform beam of length $2a$ at any point when it rests on supports at its ends, show that the bending moment at the same point, when it is clamped horizontally at both ends, under the same loads as before, is

$$M + \frac{x}{2a}(M_2 - M_1) + \tfrac{1}{2}(M_2 + M_1)$$

where M_1, M_2, denote the bending moments at the ends in the second case, and x denotes the distance from the middle of the beam.

38 (*c*). Let M denote the bending moment at any point of a uniform beam of length $2a$, loaded in any manner, when the ends rest freely on supports, and M′ the bending moment at the same point when the ends are clamped horizontally, and let x be measured from the middle of the beam.

Prove that

$$\int_{-a}^{+a} M'dx = 0; \quad \int_{-a}^{+a} M'x\,dx = 0$$

If it is given that

$$\int_{-a}^{+a} M\,dx = -A, \quad \int_{-a}^{+a} Mx\,dx = -cA,$$

show that the bending moments at the ends when the beam is clamped are

$$A\frac{a \pm 3c}{2a^2}.$$

(*A question in Honours Engineering at Manchester University asked for a proof of the final result.*)

39. A uniform beam of weight W and length $2a$ bears a concentrated load P at its middle point. Find the bending moment at distance x from the middle point, assuming that the beam rests on supports at the ends.

$$\left[\frac{W}{4a}(a \pm x)^2 - \frac{W+P}{2}(a \pm x).\right]$$

Now make use of example 38 (c) to find the bending moments at the ends when the beam is clamped there.

$$\left[\frac{a}{12}(2W + 3P).\right]$$

CHAPTER XI

POTENTIAL AND ATTRACTIONS

239. Law of Gravitation.—By considering the motion of bodies near the earth's surface, and comparing their motion with that of the moon, Newton concluded that the earth attracts all bodies with a force which varies inversely as the square of the distance of the body from the earth's centre. A study of the behaviour of the earth and the other planets led him to the great *Law of Gravitation*, which has been abundantly confirmed by all later investigations of the motions of the heavenly bodies. This is the law: *Every particle of matter attracts every other particle with a force which is proportional to the product of the masses of the two attracting particles and inversely as the square of the distance between them, and this force acts in the line joining the particles.*

Thus two particles of masses m_1 and m_2 at a distance r apart attract each other with a force $\kappa\frac{m_1 m_2}{r^2}$. Also a mass m attracts unit mass at a distance r away with a force $\kappa\frac{m}{r^2}$, κ being a constant, called the constant of gravitation, which has been found by observation. The numerical value of κ depends on the units of mass, length, and time, that we use. The most reliable values for κ have been found by measuring the actual force between two spheres in the Cavendish experiment, or rather in modern repetitions of this experiment. Measuring force in pounds weight, mass in pounds, and length in feet, the most probable value of the gravitation constant is

$$\kappa = \frac{1}{30{\cdot}2 \times 10^9} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If force is measured in poundals instead of pounds, the constant becomes

$$\kappa = \frac{1}{9{\cdot}4 \times 10^8} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If we could determine by an independent method the mean density of the earth, we could find κ from the known attraction on a body at its surface. For it will be proved later in this chapter that a spherical body such as the earth attracts external particles as if the whole of its mass were concentrated into a particle at its centre. If,

then, r is the radius of the earth in feet, the attraction on one pound mass at its surface is

$$\kappa \times \frac{\text{mass of earth}}{r^2} = \tfrac{4}{3}\pi\kappa\rho r \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where ρ is the mean density. But we know that this attraction is a force of one pound. Hence

$$1 = \tfrac{4}{3}\pi\kappa\rho r$$

Therefore
$$\kappa = \frac{3}{4\pi\rho r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

But in practice this equation is used to determine ρ and not κ. κ has to be found by such experiments as the Cavendish experiment, or by finding the attraction of a mountain of known mass on a pendulum bob, as in the Schiehallion experiment. The value of ρ obtained from (4) by using the given value of κ is about 5·6 times the density of water, that is, 5·6 × 62·4 pounds per cubic foot. The mean density of the earth may be taken roughly as $5\frac{1}{2}$ times that of water.

240. Attraction at a Point and Potential due to any Mass.—We have pointed out that the attraction of a mass m on unit mass at a distance r is $\kappa\frac{m}{r^2}$. This force on unit mass at r is called the *Attraction of the mass* m *at* r. The attraction of m on any other mass m' at r is given immediately by multiplying the attraction on unit mass by m'.

The work, dW, done on unit mass by the attraction of m in a displacement from r to $r + dr$ is

$$dW = -\kappa\frac{m}{r^2}dr \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

whence the work function, W, obtained by integrating, is

$$W = \kappa\frac{m}{r} + C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The potential energy of the unit mass due to the attraction of m is

$$V = -\kappa\frac{m}{r} - C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

It would obviously be convenient to have C zero. We therefore take C zero, and thus make the potential energy zero when the unit mass is at an infinite distance from m. Then

$$V = -\kappa\frac{m}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In dealing with the potential energy of finite bodies it is found troublesome to carry the constant κ all through our calculations; it is also a slight advantage to have a plus instead of a minus sign in (4). For these reasons we drop both the negative sign and the constant κ in the theory of potential, and only introduce them in numerical calculations. We therefore call $\frac{m}{r}$ the *Potential of* m *at distance* r.

Since the negative sign is missing it would really be more correct to call this quantity the work function. However, the student should remember that to get the potential energy of the unit mass due to any attracting masses he must multiply the potential by $-\kappa$. Thus denoting potential by V, we have

$$V = \frac{m}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and the force which m exerts on unit mass at r has a component, parallel to any line taken as x-axis, whose magnitude is

$$\kappa\frac{\partial V}{\partial x} = \kappa m\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = -\kappa m\frac{1}{r^2}\cdot\frac{\partial r}{\partial x} \quad . \quad . \quad . \quad . \quad (6)$$

If three mutually perpendicular axes be taken through the position of the attracting particle m, then, r denoting the distance of (x, y, z) from m,

$$r^2 = x^2 + y^2 + z^2$$

and therefore $$2r\frac{\partial r}{\partial x} = 2x \quad \text{and} \quad \frac{\partial r}{\partial x} = \frac{x}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Hence the force at (x, y, z) parallel to the x-axis is

$$\kappa\frac{\partial V}{\partial x} = -\kappa\frac{m}{r^2}\cdot\frac{x}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

which is obvious from a figure, because $-\kappa\frac{m}{r^2}$ is the outward force, and $\frac{x}{r}$ is the cosine of the angle between the x-axis and the line of action of the force.

Our present task is to determine the potentials and attractions of finite bodies at points fixed relative to the bodies. The potential of a finite body at any point is the sum of the potentials of its parts, and is therefore

$$V = \int\frac{dm}{r}$$

where dm is the element of mass situated at distance r from the point.

The constant κ will be equal to unity, and may therefore be omitted in potentials and attractions, if we properly choose a new unit of mass. Thus the acceleration produced in any mass by the attraction of a mass m at a distance r away is

$$f = \kappa\frac{m}{r^2} = \frac{1}{9\cdot 4 \times 10^8}\cdot\frac{m}{r^2} \text{ in foot-second units}$$

If we write M for κm this becomes

$$f = \frac{M}{r^2}$$

Let us now choose a unit of mass so that M represents the mass of the attracting body. This new unit is called the *astronomical unit of mass.* Thus we get

$$\kappa m \text{ astronomical units} = m \text{ lbs.},$$

and therefore
$$\text{one astronomical unit} = \frac{1}{\kappa} \text{ lbs.}$$
$$= 9{\cdot}4 \times 10^8 \text{ lbs.}$$

241. Potential and Attraction of a Thin Uniform Ring of Matter at a Point on its Axis.

Let a be the radius of the ring, m its mass, and x the distance of the point from the centre. Then every point of the ring is at the same distance $\sqrt{a^2 + x^2}$ from the point. Hence the potential is

$$V = \frac{m}{\sqrt{(a^2 + x^2)}} \quad . \quad . \quad . \quad (1)$$

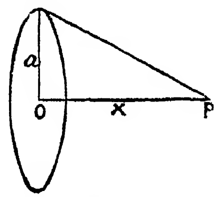


FIG. 127.

and the attraction, which is clearly along the axis, is

$$-\kappa\frac{\partial V}{\partial x} = \kappa\frac{xm}{(a^2 + x^2)^{\frac{3}{2}}} \quad . \quad . \quad (2)$$

The minus sign is introduced because $\kappa\dfrac{\partial V}{\partial x}$ is the force in the direction in which x increases, and that is the negative of the attraction.

It is necessary to point out here that the expression for V given by (1) does not enable us to find the components of the attraction perpendicular to the axis. It is true that if we differentiate V with respect to either y or z we shall get zero as the result, but this does not tell us that the force perpendicular to OP is zero; for the V given by (1) is not the complete expression for the potential at any point in space, it is only the expression which holds so long as we stick to the line OP. It is from the symmetry about the axis that we infer that there is no force perpendicular to this axis. The complete expression for V at (x, y, z) would involve y and z, and it happens that $\dfrac{\partial V}{\partial y}$ and $\dfrac{\partial V}{\partial z}$ are zero on the x-axis, but not in other positions. However y and z are involved in the full expression for V they occur as constants in (1), and therefore we cannot infer anything from this equation concerning $\dfrac{\partial V}{\partial y}$ and $\dfrac{\partial V}{\partial z}$. But since y and z are to be treated as constants in obtaining $\dfrac{\partial V}{\partial x}$, the fact that they are already in the form of constants does not affect our differentiation. Consequently we get the correct value of the force along the x-axis in $\kappa\dfrac{\partial V}{\partial x}$.

It is worth while here to get the attraction in another way.

Let dm denote an element of the ring at Q. The attraction of this element is

$$\kappa\frac{dm}{PQ^2} = \kappa\frac{dm}{x^2 + a^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and it acts along PQ.

The component of this along the axis of the ring is

$$\kappa\frac{dm}{x^2 + a^2} \times \frac{x}{\sqrt{(x^2 + a^2)}} = \kappa\frac{x}{(x^2 + a^2)^{\frac{3}{2}}}dm \quad . \quad . \quad . \quad (4)$$

The whole attraction of the ring, which is clearly the sum of the components along the axis, is

$$\kappa\frac{x}{(x^2 + a^2)^{\frac{3}{2}}}\int dm = \kappa\frac{xm}{(x^2 + a^2)^{\frac{3}{2}}} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

242. Potential and Attraction of a Thin Uniform Circular Disc at a Point on its Axis.

Let m denote the mass of the disc, a its radius, x the distance of the point from the centre of the disc.

Let the disc be divided in thin concentric rings. If y and $y + dy$ be the internal and external radii of one of these rings, its mass is

$$m \times \frac{2\pi y dy}{\pi a^2} = \frac{2m}{a^2}ydy \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Hence, by the last article, the potential of this ring is

$$\frac{2m}{a^2}ydy \times \frac{1}{\sqrt{(y^2 + x^2)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Consequently the potential of the whole disc is

$$V = \frac{2m}{a^2}\int_0^a \frac{ydy}{\sqrt{(y^2 + x^2)}}$$

$$= \frac{2m}{a^2}\{\sqrt{(a^2 + x^2)} - x\}$$

Hence the attraction is

$$-\kappa\frac{dV}{dx} = \frac{2m\kappa}{a^2}\left\{1 - \frac{x}{\sqrt{(a^2 + x^2)}}\right\}$$

$$= \frac{2m\kappa}{a^2}(1 - \cos\theta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where θ is the angle subtended by a radius of the disc at the point on the axis.

If ρ is the surface density of the disc, that is, the mass per unit area, we may write the attraction F in the form

$$F = 2\pi\kappa\rho(1 - \cos\theta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

since $m = \pi\rho a^2$

Equation (4) shows that if ρ is the same for different discs, the attraction will be the same at such points on their axes that the radii subtend equal angles at those points.

243. Attraction of an Infinite Plate or Disc with Uniform Surface Density.

If we make a infinite in the expression for V in the last article, we get an infinite value for V. But the attraction is not infinite when a is infinite. The attraction for an infinite disc or plate is

$$F = 2\pi\kappa\rho \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

a quantity which is independent of x. Besides, since any line perpendicular to the plane may be considered as the axis of an infinite plate, it follows that the attraction everywhere outside the plane itself is the constant quantity $2\pi\kappa\rho$.

Although infinite plates do not exist, yet the equation (4) of the last article shows that any plane of matter may be regarded as infinite in finding the component attraction perpendicular to the plate at a point such that θ is nearly a right angle. Moreover, the plane need not be circular. If a circular piece can be cut out of it such that the point lies on the axis of the circle and θ is nearly a right angle, the plane may still be treated as an infinite disc, for the component attraction of the matter outside the circle perpendicular to the plate will be negligible, although the component parallel to the plane may not be negligible.

The attraction of an infinite plate, with a circular hole cut out, at a point on the axis of the circle, is the attraction of the infinite plate minus that of the circle, and is therefore $2\pi\kappa\rho \cos \theta$.

244. Attraction of a Homogeneous Solid Cylinder at a Point on its Axis.

Let ρ be the density of the cylinder, a its radius, l its length; and let d denote the distance of the point P, at which the attraction is required, from the nearer face. Thus PE $= d$.

ABN is a section of the cylinder containing the axis CE. The cylinder may be supposed to be generated by rotating this rectangle about CE. Then the small strip MCN generates a circular disc. Let PC$=x$, CD$=dx$.

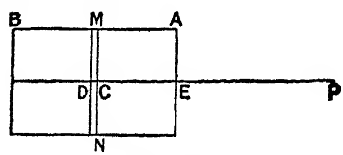


FIG. 128.

The mass of this disc is $\rho a^2 dx$. Then, by Art. 242, the attraction of the disc at P is

$$2\pi\kappa\rho dx\left\{1 - \frac{x}{\sqrt{(a^2+x^2)}}\right\} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Therefore the attraction of the whole cylinder is

$$2\pi\kappa\rho\int_d^{d+l}\left\{1 - \frac{x}{\sqrt{(a^2+x^2)}}\right\}dx$$

$$= 2\pi\kappa\rho\{l - \sqrt{(l+d)^2+a^2} + \sqrt{d^2+a^2}\}$$

$$= 2\pi\kappa\rho(l - \text{PB} + \text{PA}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If l were infinite PB would also be infinite, and such that $(l - \text{PB})$ would be $-d$. Consequently the attraction of an infinite cylinder is

$$2\pi\kappa\rho\,.\,(\text{PA} - d)$$

245. Attraction of a Uniform Rod, bent into the Form of a Circular Arc, at the Centre of the Circle of which it forms a Part.

Let ρ be the mass of unit length of the rod, and let M be the mid-point of the rod. Let angle MOB $= a$, angle MOQ $= \theta$, angle QOQ$' = d\theta$. If r is the radius, the mass of QQ$'$ is $\rho r d\theta$, and its attraction is

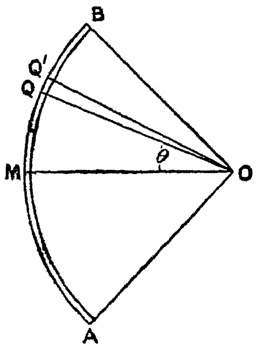


FIG. 129.

$$\kappa\frac{\rho r d\theta}{r^2} = \kappa\frac{\rho}{r}d\theta$$

From symmetry it is clear that the resultant attraction acts along MO. Now the component of the attraction of QQ$'$ along MO is

$$\kappa\frac{\rho}{r}d\theta\,.\cos\theta$$

Hence the resultant attraction of the rod is

$$\text{F} = \kappa\frac{\rho}{r}\int_{-a}^{+a}\cos\theta d\theta = \kappa\frac{2\rho}{r}\sin a \quad (1)$$

Since the whole length of the rod is $2ra$, and therefore the total mass m is $2ra\rho$, we may write the attraction thus

$$\text{F} = \kappa\frac{m}{r^2}\cdot\frac{\sin a}{a} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

The potential at O is obviously $\frac{m}{r}$, but we cannot get the attraction from this, since it is just the potential at a single point, and not a general expression for potential at any point. If we were to find the potential at a point on MO at a distance x from O, we could, by differentiating this expression with respect to x, find the attraction at x. But to find the attraction at O the preceding method of direct integration is much easier.

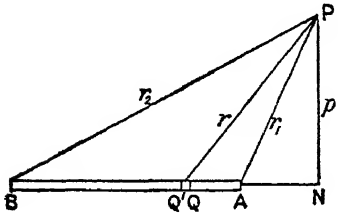


FIG. 130.

246. Potential of a Thin Uniform Rod of Length l at any Point P.

Let m be the mass of the rod. In the figure, let NQ $= x$, NA $= x_1$, NB $= x_2$, QQ$' = dx$. The mass of QQ$'$ is $\frac{m}{l}dx$. Hence its potential at P is $\frac{m}{l}\cdot\frac{dx}{r}$. Therefore the potential of the whole rod is

$$V = \frac{m}{l}\int_{x_1}^{x_2}\frac{dx}{r} = \frac{m}{l}\int_{x_1}^{x_2}\frac{dx}{\sqrt{(p^2+x^2)}}$$

$$= \frac{m}{l}\left[\log\{(p^2+x^2)^{\frac{1}{2}}+x\}\right]_{x_1}^{x_2}$$

$$= \frac{m}{l}\log\frac{\sqrt{(p^2+x_2^2)}+x_2}{\sqrt{(p^2+x_1^2)}+x_1}$$

$$= \frac{m}{l}\log\frac{r_2+x_2}{r_1+x_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now $$r_2^2 - x_2^2 = p^2 = r_1^2 - x_1^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Therefore $$\frac{r_2+x_2}{r_1+x_1} = \frac{r_1-x_1}{r_2-x_2}$$

$$= \frac{r_2+x_2+r_1-x_1}{r_1+x_1+r_2-x_2}$$

$$= \frac{r_1+r_2+l}{r_1+r_2-l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Hence $$V = \frac{m}{l}\log\frac{r_1+r_2+l}{r_1+r_2-l} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If an ellipse be drawn through P with A and B as foci, the sum of the focal distances of all points on this ellipse is the same as for P, namely $(r_1 + r_2)$. Hence the potential at all points on this ellipse will be the same as the potential at P. Thus the potential is constant over this ellipse, and therefore also over the ellipsoid obtained by revolving this ellipse about AB.

247. Attraction of a Thin Rod.—The attraction may be found by differentiating the potential with respect to two perpendicular co-ordinates, and finding the resultant force. But it is easier to get the attraction without using the potential.

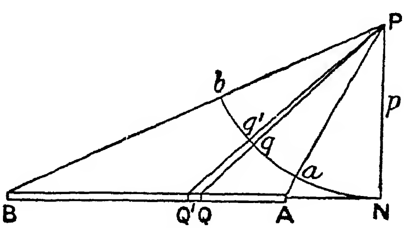


FIG. 131.

As in the last article, QQ′ is an element of the rod. A circle is drawn with centre P and radius p cutting PA, PB, PQ, PQ′, in a, b, q, q', respectively.

Let angle NPQ $= \theta$, angle QPQ′ $= d\theta$. Then the attraction of QQ′ is

$$\kappa\frac{m}{l}\cdot\frac{QQ'}{PQ^2} = \kappa\frac{m}{l}\cdot\frac{dx}{PQ^2}$$

But $$x = p\tan\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and therefore $$dx = p\sec^2\theta d\theta = p\frac{PQ^2}{p^2}d\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence the attraction of QQ′ is

$$\kappa\frac{m}{l}\cdot\frac{p}{p^2}d\theta = \kappa\frac{m}{l}\cdot\frac{qq'}{p^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Now if a rod of the same material and thickness as the given rod be bent to fit the circular arc *ab*, the mass of the portion *qq′* would be $\frac{m}{l}qq'$, and its attraction would be exactly the same as we have found in (3) above, and it would act in the same straight line QP. It follows, therefore, that the attraction of the given rod at P is the same in all respects as the attraction of a similar (but shorter) rod lying along *ab*. This attraction bisects the angle APB, and, by Art. 245, its magnitude is given by

$$F = \kappa\frac{2m}{pl}\sin\frac{\text{APB}}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

248. Attraction of a very Long Rod.—If AB is very long compared with p, and N lies between A and B at a great distance from each end, then the angle APB is nearly two right angles, and the bisector of this angle is nearly perpendicular to the rod. Consequently the attraction in this case is nearly perpendicular to the rod, and its magnitude is nearly

$$F = 2\kappa\frac{m}{pl}\sin\frac{\pi}{2} = 2\kappa\frac{\rho}{p}$$

where ρ is the mass of unit length. If A and B are at opposite infinities this result is exactly true.

249. Components of the Attraction of a Thin Rod perpendicular and parallel to the Rod.

Let the angles NPA, NPB, and APB, be denoted by α, β, and γ. Then the component of F (which acts along the bisector of the angle APB) parallel to BA is

$$F\sin\left(\alpha+\frac{\gamma}{2}\right) = 2\kappa\frac{m}{pl}\sin\frac{\gamma}{2}\sin\left(\alpha+\frac{\gamma}{2}\right)$$

$$= \kappa\frac{m}{pl}(\cos\alpha - \cos\beta) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or

$$= \kappa\frac{m}{l}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

because $p = r_1\cos\alpha = r_2\cos\beta$.

Also the component of F perpendicular to the rod is

$$F\cos\left(\alpha+\frac{\gamma}{2}\right) = 2\kappa\frac{m}{pl}\sin\frac{\gamma}{2}\cos\left(\alpha+\frac{\gamma}{2}\right)$$

$$= \kappa\frac{m}{pl}(\sin\beta - \sin\alpha) \quad . \quad . \quad . \quad . \quad . \quad (3)$$

When the attraction of a rod at a point in its length is required the expression given in (2) of this article should be used, or this expression

can be easily obtained by direct integration for this particular case. But equation (4) of Art. 247 is not of much use for this case, because when the point is on the rod, both p and the angle APB are zero, and it is therefore necessary to find a limiting value.

250. Potential and Attraction of a Thin Spherical Shell of Uniform Density.

Case I. Point outside the shell.

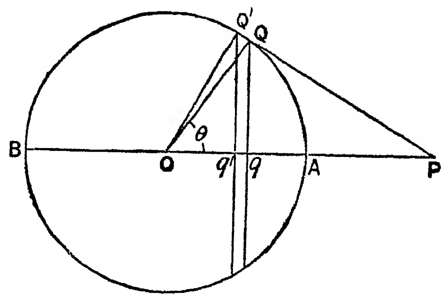


FIG. 132.

P is the point at which the potential is required; q, q', are two very near points on the diameter through P; qQ, $q'Q'$, are perpendicular to OP and are both in the same plane.

Let $Pq = z$, $PQ = r$, $qq' = dz$, $OP = x$, angle $QOQ' = d\theta$. Also let ρ be the mass of unit area, and a the radius of the sphere.

The mass of the spherical shell between the two planes through q and q' perpendicular to OP is

$$\rho \cdot 2\pi \cdot Qq \cdot QQ' = \rho \cdot 2\pi a \sin\theta \cdot a d\theta = \rho \cdot 2\pi a dz \quad . \quad . \quad (1)$$

because $z = x - a\cos\theta$, and therefore $dz = a\sin\theta d\theta$.

The potential of this mass at P is

$$\rho \cdot 2\pi a \frac{dz}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

But
$$a^2 = r^2 + x^2 - 2zx \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Hence
$$2rdr = 2xdz \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Therefore the potential of the thin strip is

$$2\pi\rho a \cdot \frac{rdr}{xr} = 2\pi\rho a \cdot \frac{dr}{x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Consequently the potential of the whole shell is

$$V = \frac{2\pi\rho a}{x}\int_{PA}^{PB} dr = \frac{2\pi\rho a}{x} \cdot (PB - PA)$$

$$= \frac{4\pi\rho a^2}{x}$$

$$= \frac{\text{mass of shell}}{x} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This is exactly the same as if the whole mass were concentrated at the centre of the shell.

The attraction is

$$-\kappa\frac{dV}{dx} = \kappa\frac{4\pi\rho a^2}{x^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Thus a spherical shell of uniform density attracts external particles, and therefore all external bodies, exactly as if the mass of the shell were placed at its centre.

Case II. If P is inside the shell, everything, as far as equation (5) above, is just the same. In this case we get

$$V = \frac{2\pi\rho a}{x}.(PB - PA) \quad . \quad . \quad . \quad . \quad . \quad (8)$$

just as before. But it might seem as if we ought to take PB (or PA) as a negative quantity. But PA and PB are the initial and final values of PQ, a quantity which never passes through zero. Consequently PB — PA is the actual difference of the lengths, both regarded as positive quantities. Hence

$$V = \frac{2\pi\rho a}{x}.2x = 4\pi\rho a \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

for an internal point.

Thus the potential at an internal point is constant, that is, it does not depend on x. Consequently the attraction, which is proportional to $\frac{dV}{dx}$, is zero everywhere inside the shell.

251. Attraction of a Solid Sphere and of a Hollow Sphere.—If a solid sphere or a hollow sphere is composed of uniform shells such as we considered in the last article, such a sphere will attract all external particles exactly as if its mass were concentrated at its centre. And a particle inside the mass of the sphere will be attracted by all the shells to which it is external, and will experience no force from the other shells. We will express this result in symbols.

If the density of a sphere at distance a from its centre is a function of a only, the sphere can be divided into uniform thin shells. Let ρ denote the density at distance a. This density ρ need not be a continuous quantity; it may be zero for some values of a, and it may change its value suddenly as a increases gradually.

The mass of a thin shell of thickness da is $4\pi\rho a^2 da$, and its attraction on an external particle at distance r from its centre is

$$\kappa\frac{4\pi\rho a^2 da}{r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The attraction of the whole sphere at the point is therefore

$$F = \kappa\frac{4\pi}{r^2}\int_0^r \rho a^2 da \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and this applies whether the particle is inside the mass or not. But if the particle is outside the mass, ρ will be zero from the point where a is equal to the radius of the bounding sphere of the mass up to $a = r$. In this case, if $b =$ the radius of the boundary, the preceding value for F is the same as

$$F = \kappa\frac{4\pi}{r^2}\int_0^b \rho a^2 da \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Suppose we are dealing with a solid sphere of uniform density, and we want the attraction inside the mass at a distance r from the centre. The mass inside the sphere of radius r (which is the only mass that exerts a resultant attraction at the point) is $\frac{4}{3}\pi\rho r^3$. The attraction of this at r is

$$F = \kappa \cdot \tfrac{4}{3}\pi\rho r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Thus the attraction of a uniform solid sphere at a point inside its mass is proportional to the distance from the centre.

The result given by equation (2) agrees, of course, with that in (4).

252. Potential and Attraction of a Uniform Plate at a Point in its Plane not occupied by Matter.

Let PQR, pqr, be the boundaries of the plate, and O the point at which the attraction is required. The whole area can be divided into such strips as PQqp by lines through O. We must first find the attraction and potential of such a strip as PQqp separately.

In Fig. 133B, let the angle $P'O'Q' = \phi$ a very small angle. Let M, N,

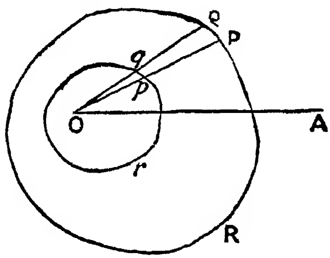


FIG. 133A.

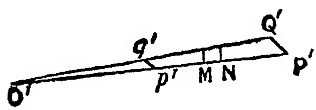


FIG. 133B.

be two neighbouring points on $p'P'$, and let $O'M = r$, $MN = dr$. The strip cut out by lines through M and N perpendicular to O'P' is approximately a rectangle with sides dr and $r\phi$. If ρ is the mass of unit area, the mass of this strip is $\rho r\phi dr$. Hence the attraction at O' is

$$\kappa\frac{\rho r\phi dr}{r^2} = \kappa\rho\phi\frac{dr}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and the potential at O' is $\rho\phi dr$ . . . . (2)

If $O'p' = r_1$, $O'P' = r_2$, the attraction of the strip is

$$\kappa\rho\phi\int_{r_1}^{r_2}\frac{dr}{r} = \kappa\rho\phi\log\frac{r_2}{r_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and the potential is

$$\rho\phi\int_{r_1}^{r_2}dr = \rho\phi(r_2 - r_1) \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Now returning to our plate, let OA be any fixed line in the plane

of the plate, and let angle AOP $= \theta$, angle POQ $= d\theta$. Then the potential of the strip PQqp at O is

$$\rho(r_2 - r_1)d\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and the attraction is

$$\kappa\rho \log \frac{r_2}{r_1} d\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The potential of the whole plate at O is therefore

$$V = \rho\int(r_2 - r_1)d\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

When the polar equations of the bounding curves are known, this integral is an integral of a function of θ between known limits. For the figure shown the limits of θ would be 0 and 2π; but these will not be the limits for all plates, because all plates would not completely encircle O.

The component attraction parallel to AO of the strip PQqp is

$$\kappa\rho \log \frac{r_2}{r_1} \cos\theta \, d\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Integrating for the whole plate, the attraction parallel to AO is

$$X = \kappa\rho\int \log \frac{r_2}{r_1} \cos\theta \, d\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Similarly the component attraction perpendicular to OA is

$$Y = \kappa\rho\int \log \frac{r_2}{r_1} \sin\theta \, d\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

We will apply our results to find the attraction and potential of a sector of a disc bounded by concentric circles at the centre of the circles.

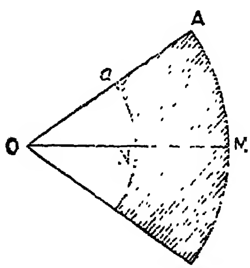


FIG. 134.

For such a disc r_1 and r_2 are constant. If 2α is the angle of the sector, the potential at the centre is obviously

$$V = 2\rho\alpha(r_2 - r_1) \quad . \quad . \quad (11)$$

The resultant attraction is along the bisector of the angle, and if θ be measured from this bisector OM, the magnitude of the attraction is

$$F = \kappa\rho\int_{-\alpha}^{+\alpha} \log \frac{r_2}{r_1} \cos\theta \, . \, d\theta$$

$$= 2\kappa\rho \sin\alpha \log \frac{r_2}{r_1} \quad . \quad . \quad (12)$$

The attraction is of the same form for any plate for which $\frac{r_2}{r_1}$ is constant, and moreover the attraction acts along the bisector of the angle at O. For, if we measure θ from this bisector, equation (10) shows that there is no force perpendicular to the bisector when $\frac{r_2}{r_1}$ is cons ant, and

equation (12) gives the attraction along the bisector. Also, if α is two right angles the attraction is zero. Thus if a hole be cut in any uniform plate, the boundary of the hole being a curve similar and similarly situated with the boundary of the plate, the attraction of the remainder will be zero at the centre of similitude of the two curves provided there is matter all round the hole, because for such a plate $\frac{r_2}{r_1}$ is constant and $\alpha = \pi$. For instance, in the plate just considered, suppose the centre of the circular boundaries of the hole had been somewhere in NM, and suppose the hole had had the same shape as the plate has in the example. Then the boundaries of the hole and the plate from which it is cut would be similar and similarly situated curves, with the centre of similitude somewhere inside the hole and in NM. The attraction at this centre of similitude would therefore be zero.

253. Gauss's Theorem.—Before proceeding to this theorem we will explain what is meant by a solid angle.

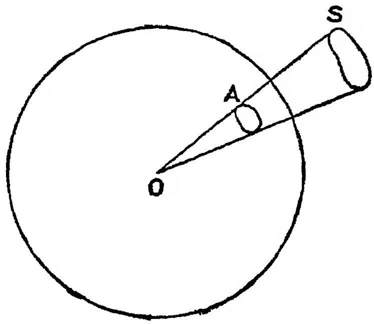


FIG. 135.

If every point of a closed surface S be joined to the centre of a sphere of radius r, an irregular cone will be thus formed which will meet the surface of the sphere in a closed curve. Then the solid angle subtended at O by the area S is defined to be $\frac{A}{r^2}$, where A is the area of the closed curve on the surface of the sphere. This quantity is the same whatever sphere we take with centre O, because the intersection of the cone with different spheres will give similar curves, whose areas will clearly be proportional to the square of their linear dimensions—that is, proportional to r^2. Hence $\frac{A}{r^2}$ will be the same for all spheres with centre O.

The preceding definition corresponds to the definition of the radian measure of a plane angle. In plane geometry lines take the place of surfaces, and a circle takes the place of the sphere.

The solid angle subtended at the centre by any spherical surface is the sum of the solid angles subtended by its parts. Thus we might divide it into two halves, or eight octants. Then the solid angle is clearly

$$\frac{\text{area of sphere}}{(\text{radius})^2} = 4\pi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

We will now state and prove Gauss's Theorem.

If, in any region where attracting matter is situated, any closed geometrical surface be taken and divided into infinitely small elements of area, and each element of area be multiplied by the component, along

the inward normal to the surface at that point, of the attraction due to all the matter in the region, then the sum of these products is equal to the product of $4\pi\kappa$ and the mass enclosed by the surface.

Let dS denote a small element of area on our geometrical surface, and let N be the component of the attraction along the normal to dS drawn towards the inside of the surface. Then Gauss's theorem states that

$$\int N dS = 4\pi\kappa \times (\text{mass enclosed by surface}) \quad . \quad . \quad . \quad (2)$$

the integral being taken over the whole of the closed surface.

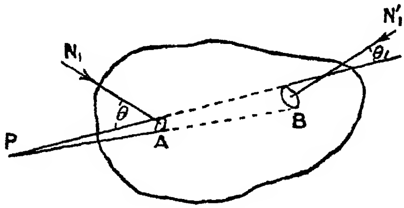


FIG. 136.

We will first deal with single particles.

Let a particle m_1 be situated outside the closed surface at a point P. Let a small cone be drawn through P to meet the surface in A and B, and let dS and dS' denote the elements of area cut out by the cone at A and B. Let N_1 and N_1' be the normal components of the attraction of m_1 at these points. θ and θ_1 are the angles between the normals at A and B and the thin cone PAB as shown in the figure.

The attraction at A is $\dfrac{m_1}{PA^2}$, and the component of this along the *inward* normal is

$$N_1 = -\kappa\frac{m_1}{PA^2} \cdot \cos\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Hence
$$N_1 dS = -\kappa\frac{m_1}{PA^2} dS \cos\theta$$

Now $dS \cos\theta$ is the area of the normal section of the cone at A, that is, the area of the section perpendicular to PA. Hence

$$\frac{dS\cos\theta}{PA^2} = d\omega, \text{ the solid angle of the cone} \quad . \quad . \quad (4)$$

Hence
$$N_1 dS = -\kappa m_1 d\omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

In the same way we can prove that

$$N_1' dS' = +\kappa m_1 d\omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The sign is positive in (6) because N_1' is clearly positive. In fact, the component along the inward normal is positive wherever the cone passes out of the region enclosed by the surface, and negative where it passes in. Now

$$N_1 dS + N_1' dS' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The whole surface can clearly be divided into pairs of elements such as those at A and B, both in the same small cone; and since for every

such pair an equation similar to (7) would be true, it follows that, for the whole surface,

$$\int N\,dS = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

for this equation is obtained by adding together all such equations as (7).

Now let us turn to an internal particle.

Let M_1 be the mass of an internal particle. At A the cone passes out of the surface, and therefore

$$N_1 dS = \kappa M_1 d\omega$$

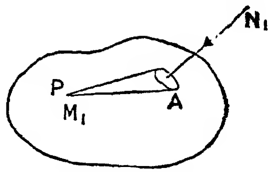


Fig. 137.

Summing for the whole surface

$$\int N\,dS = \kappa M_1 \int d\omega$$
$$= 4\pi\kappa M_1 \quad . \quad . \quad . \quad (9)$$

It is clear that $\int d\omega$ for a closed surface is the same as for a sphere with centre P, and this we know to be 4π.

Now suppose there are particles M_1, M_2, M_3, etc., inside the surface, and let the normal components of the attractions of these at an element dS be N_1, N_2, N_3, etc. Also suppose there are particles m_1, m_2, m_3, etc., outside the surface, and let the normal components of the attraction of these at dS be n_1, n_2, n_3, etc. Let N denote the normal component of the attraction of all these particles. Then clearly

$$N = (N_1 + N_2 + N_3 + \ldots) + (n_1 + n_2 + n_3 + \ldots) \quad . \quad (10)$$

Hence

$$\int N\,dS = \int (N_1 + N_2 + N_3 \ldots)dS + \int (n_1 + n_2 + n_3 + \ldots)dS$$
$$= 4\pi\kappa(M_1 + M_2 + M_3 + \ldots) + 0$$
$$= 4\pi\kappa \times (\text{total mass inside surface}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Since this result is true for any number of particles it is true for any finite body, since it may be considered to be composed of a very large number of particles. It may happen that, for a particular geometrical surface, the cones we have considered meet the surface in more points than we have assumed. Thus the cone for an internal particle may meet at three points, as shown in Fig. 138. But the additional number of intersections is in all cases an even number, because every time that the cone enters the region enclosed by the surface it will have to be produced till it leaves the region again. Thus the additional intersections occur in pairs, one on entering and one on leaving. Now, exactly as in dealing with an external particle, we can prove that the sum of the contributions of the elements B and C to the integral

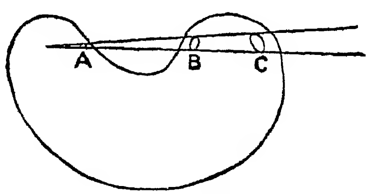


Fig. 138.

$$\int N\,dS$$

is zero. The value of the integral is, therefore, just the same as if we

did not produce the cone beyond A; that is, the cone need not be produced when once it has got outside the surface. The additional intersections do not therefore affect our results, and Gauss's theorem remains true for any kind of closed surface.

The same law of force holds for electricity as for ordinary matter, except that the force is repulsive between two positive charges instead of attractive as with positive matter. If we take the force along the outward normal instead of along the inward normal, Gauss's theorem will be equally true for electricity. The total mass inside the surface will, in this case, correspond to the algebraic sum of the charges. Also κ will be a different constant for electricity.

The quantity $\mathrm{N}d\mathrm{S}$ is called the *normal induction* at the element $d\mathrm{S}$, and $\int \mathrm{N}d\mathrm{S}$ is called the total normal induction over the surface.

254. Applications of Gauss's Theorem.—There are many useful applications of Gauss's theorem, and particularly in electrostatics. We will give here a few examples.

(i) To find the attraction of a uniform spherical shell, or of any number of concentric uniform spherical shells, at a point P at distance r from the centre, let a sphere be taken through P concentric with the shells. Applying Gauss's theorem to this sphere

$$\int \mathrm{N}d\mathrm{S} = 4\pi\kappa \times (\text{mass inside sphere through P}) \quad . \quad . \quad (1)$$

But N is constant all over the spherical surface, and it is equal to the resultant attraction F. Hence

$$\int \mathrm{N}d\mathrm{S} = \mathrm{F}\int d\mathrm{S} = 4\pi r^2 \mathrm{F} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Therefore by (1) and (2)

$$\mathrm{F} = \kappa\frac{\mathrm{M}}{r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where M is the total mass inside the sphere through P. This has already been proved in Art. 251.

(ii) *Attraction of an infinite uniform plane at any point.*

It is obvious that the attraction of a plane infinite in all directions has no component parallel to the plane. To find the attraction at P take, as Gauss surface, a thin cylinder with axis PP′ perpendicular to the plane, P and P′ being at equal distances from the plane on opposite sides of it. Let F be the attraction at P, and therefore at P′. Since there is no force parallel to the plane, the normal component of the attraction all over the curved surface of the cylinder is zero. Consequently the total normal induction over the surface of the cylinder is merely the normal induction over the ends; that is, 2FA, where A is the area of the section of the

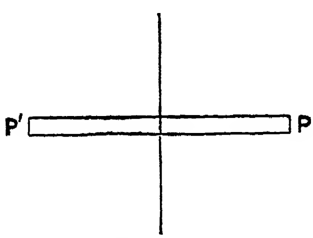


FIG. 139.

cylinder. Applying Gauss's theorem to this cylinder, we get, if ρ is the mass per unit area on the plane,

$$2FA = 4\pi\kappa . \rho A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

whence
$$F = 2\pi\kappa\rho \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

which agrees with equation (1), Art. 243.

(iii) *Attraction of an infinite rod at distance* p *from the rod.*

Let ρ be the mass of unit length. Let us now apply Gauss's theorem to a cylinder with the rod as axis and radius p. The normal induction over the plane faces of the cylinder is zero, because there is no component attraction parallel to the rod. Also the normal force over the curved surface is perpendicular to that surface; that is, $N = F$, the whole attraction. Hence, if l is the length of the cylinder, Gauss's theorem gives

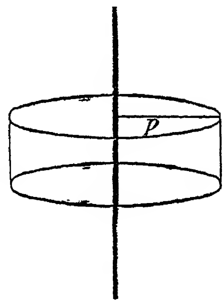


FIG. 140.

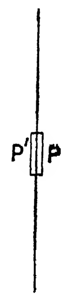
$$2\pi plF = 4\pi\kappa . \rho l \quad . \quad . \quad . \quad (6)$$

Consequently
$$F = 2\kappa\frac{\rho}{p} \quad . \quad . \quad . \quad . \quad (7)$$

the same result as in Art. 248.

255. The normal component of the attraction of any continuous distribution of matter on a plane at a point infinitely near the plane.

Let P be the point at which the attraction is required. The attraction will clearly be the same at P′ at the same distance on the other side of the plane. Let us therefore take a cylinder with the plane faces through P and P′, and the generating lines perpendicular to the plane. Since P and P′ are infinitely near the plane, the area of the curved portion of this cylinder can be made infinitely small compared with the area of the ends without making the ends themselves large. We may therefore neglect the normal induction of the curved surface in comparison with that over the ends in applying Gauss's theorem to this cylinder.



Let ρ be the mass per unit area in the neighbourhood of P, and let N be the normal component of the attraction over the ends of the cylinder. Then Gauss's theorem gives

$$2NA = 4\pi\kappa\rho A;$$

whence
$$N = 2\pi\kappa\rho \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

FIG. 141.

This result could have been obtained by considering the attraction of the mass in the neighbourhood of P to be the same as that of an infinite plane with the same density. For, equation (4), Art. 242, gives for the attraction of a disc at a point on its axis

$$F = 2\pi\kappa\rho(1 - \cos\theta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where 2θ is the angle subtended by a diameter of the disc at the point. If θ is nearly a right angle this gives

$$F = 2\pi\kappa\rho \text{ nearly} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Now θ will be nearly a right angle for a point infinitely near even a small disc. Consequently, a small circular portion of the plane with P on its axis may be considered as an infinite disc in finding the attraction at P. The attraction of this disc is therefore $2\pi\kappa\rho$.

Now the component, perpendicular to the plane, of the attraction of the rest of the plane is very small, because the line joining P to any particle in this remainder (which is the direction of the attraction of that particle) is very nearly parallel to the plane. Thus the normal component of the attraction is the attraction of the small disc in the neighbourhood of P, which we have found to be $2\pi\kappa\rho$.

256. The alteration in the attraction of any surface distribution of matter on crossing the surface.

Exactly as in the last article, where we dealt with a plane distribution of matter, a small portion of the surface surrounding any point A may be regarded as an infinite plane in finding the attraction at a point P infinitely near the surface on the normal through A. On crossing the surface the direction of this attraction is reversed, and consequently the alteration in the attraction of this small area is twice the attraction on either side. If ρ is the surface density in the neighbourhood of the point, the attraction at P of this small portion is $2\pi\kappa\rho$, and consequently the alteration in the attraction is $4\pi\kappa\rho$. The attraction of the remainder of the body does not appreciably alter in passing from a point infinitely near the surface on one side to a point infinitely near on the other. Hence the whole alteration in the attraction is $4\pi\kappa\rho$ along the normal to the surface through the point at which the surface is crossed.

For example, in crossing the surface of a uniform spherical shell the change in attraction is $4\pi\kappa\rho$ along the radius. This agrees with our previous results, because the attraction just outside the sphere is

$$\frac{4\pi\kappa\rho a^2}{a^2}=4\pi\kappa\rho$$

and the attraction just inside is zero.

257. We shall now prove a few general theorems on potential. It will be necessary, first of all, to define some of the terms we shall use.

An *Equipotential Surface* for any attracting matter is a surface over which the potential is constant.

A *Line of Force* is a line whose tangent at every point of its length is in the direction of the attraction at that point.

A *Tube of Force* is a tube formed by lines of force drawn through every point on any closed curve.

A *Filament* is an infinitely thin tube of force.

If the attracting mass is a thin rod it has been proved that the equipotential surfaces are a system of ellipsoids of revolution having their foci at the ends of the rod. It was also proved that the attraction at any point acts along the bisector of the angle between the lines joining the point to the ends of the rod. But, by a property of the ellipse, this bisector is perpendicular to the ellipse through the attracted point which has its foci at the ends of the rods. But a line which meets a

system of confocal ellipses at right angles is an hyperbola with the same foci. Thus the equipotential surfaces are a system of confocal ellipsoids of revolution, and the lines of force are everywhere perpendicular to the equipotential surfaces, and they are a system of confocal hyperbolas in every plane containing the rod.

258. The line of force through any point is perpendicular to the equipotential surface through that point.

The work done on unit mass by the attracting forces is $\kappa . \delta V$, where δV is the alteration in the potential in the displacement considered. Now let P and Q be two neighbouring points on an equipotential surface. In going from P to Q the change of potential is zero, and consequently the work done is zero. Let δs be the length of the straight line joining P and Q. Now the mean force along PQ is

$$\kappa \frac{\delta V}{\delta s} = 0 \text{ absolutely} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Hence, when δs is infinitely small,

$$\kappa \frac{dV}{ds} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

that is, the force along a tangent to the equipotential surface is zero. This is true for any tangent line to the surface. Since, then, the attraction has no component along any line in the equipotential surface, the resultant attraction must act along the normal to the surface. Thus the line of force through P is normal to the equipotential surface through P.

259. If two very near equipotential surfaces be drawn, the force at any point between them is inversely proportional to the normal distance between the surfaces at that point.

Let δV be the difference of potential on the two surfaces, F the resultant force at any point P between the surfaces, δn the length of the line of force through P intercepted by the surfaces.

Then the work done on unit mass as it is moved from the surface of lower to the surface of higher potential is $\kappa\delta V$. But if the mass be moved along the line of force through P, this work is also $F\delta n$. Hence

$$F\delta n = \kappa\delta V \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and therefore

$$F = \frac{1}{\delta n} . \kappa\delta V \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

But δV is the same for all positions of P, because it is the difference of potential between two equipotential surfaces. Hence at different points $F \propto \frac{1}{\delta n}$.

Thus, if a series of equipotential surfaces be drawn, by merely comparing the closeness of the surfaces at different points we can make a comparison between the attraction at these points; the closer the surfaces the greater the attraction.

260. The force at different points of the same filament varies inversely as the area of the normal section of the filament.

Let P and Q be two points on the same filament, F and F′ the forces at these points, σ and σ' the areas of the normal sections at P and Q.

Then applying Gauss's theorem to the closed surface formed by the normal sections at P and Q and the filament between these sections we get

$$F\sigma - F'\sigma' = 0$$

which proves the theorem.

The signs are different because the forces act one along the inward, and the other along the outward, normal.

261. Two equipotential surfaces at different potentials cannot intersect.

If two such surfaces did intersect there would be two different potentials at the line of intersection. But this is impossible because the potential at any point is a single-valued function of the co-ordinates of the position. It is clearly not possible for the potential of a single particle m, namely $\frac{m}{r}$, to have two different values at the same point. It follows, therefore, that the potential of any number of particles can only have one value at any point. Hence two different equipotential surfaces cannot meet.

262. If a closed equipotential surface contains no mass, the potential inside the surface is constant and equal to its value at the surface.

If the potential is not constant within the given surface, then there is a system of closed non-intersecting equipotential surfaces within the given surface. Let Q be a point on any one of these surfaces. Now it is clear that, since the equipotential surface at which the potential is greater than at Q lies entirely inside or entirely outside the surface through Q, the force at this latter surface is everywhere along the inward normal or everywhere along the outward normal, for the direction of the force is towards the surface of greater potential. That is, the component force N along the inward normal to the surface through Q has everywhere the same sign. But by Gauss's theorem

$$\int N\,dS = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

the integral being taken over this surface. Now since N has always the same sign, and dS cannot be negative, the only way in which (1) can be satisfied is by N being zero all over the surface. But the normal component to an equipotential surface is the resultant force. It follows, therefore, that there is no force inside the given surface. Hence no work could be done on a particle moving about in the interior of the surface; that is, the work function, and consequently the potential, is constant inside the surface.

The preceding argument verifies what we have already proved by direct calculation of the attraction, namely, that the force inside a hollow sphere is zero.

The result proved in this article is very useful in electrostatics. When a hollow conductor is charged with electricity the outer surface of the conductor is an equipotential surface, and the charge is on this surface. The electric charge, therefore, exerts no force inside the conductor, and the potential throughout the interior is equal to that at the surface.

263. Mutual Potential.—It may be proved by the method of Art. 240 that the work done by the mutual attraction of two particles as they move from an infinite distance apart to a distance r apart is

$$\kappa\frac{m_1 m_2}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and this is the same whether only one particle or both particles move. The potential energy of the two particles may, therefore, be taken as

$$-\kappa\frac{m_1 m_2}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

But for the same reasons as we called $\frac{m}{r}$ the potential of m, we shall call $\frac{m_1 m_2}{r}$ the *Mutual Potential* of the two particles.

Now if three particles m_1, m_2, m_3, are at distances r_{12}, r_{23}, r_{31}, apart (the suffixes indicate which particles the distances refer to), the work done by their mutual attractions in coming from an infinite distance apart is

$$\kappa\left(\frac{m_1 m_2}{r_{12}} + \frac{m_2 m_3}{r_{23}} + \frac{m_3 m_1}{r_{31}}\right) \quad . \quad . \quad . \quad . \quad . \quad (3)$$

For, the work done in bringing m_1 and m_2 into position and leaving m_3 at infinity is

$$\kappa\frac{m_1 m_2}{r_{12}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and the work done on m_3 by the other two particles as m_3 is brought from infinity is the sum of the works done by each particle separately, namely,

$$\kappa\frac{m_1 m_3}{r_{13}} + \kappa\frac{m_2 m_3}{r_{23}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Thus the expression (3) is the total work done.

We shall call the expression

$$W = \frac{m_1 m_2}{r_{12}} + \frac{m_2 m_3}{r_{23}} + \frac{m_3 m_1}{r_{31}} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

the mutual potential of the three particles.

Now the potential at the position of m_1, due to m_2 and m_3, is

$$V_1 = \frac{m_2}{r_{12}} + \frac{m_3}{r_{13}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Hence

$$V_1 m_1 = \frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Consequently, if V_2 and V_3 are defined similarly to V_1, we get

$$V_1m_1 + V_2m_2 + V_3m_3 = 2W \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Writing ΣVm for the expression on the left of equation (9), we may write the value of the mutual potential of the three particles thus

$$W = \tfrac{1}{2}\Sigma Vm \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The preceding investigation can be extended to any number of particles and equation (10) will be true in all cases. V, it must be noticed, is the potential at m of all the particles except m. We shall show in the next article that it makes no difference if we take V as the potential of all the particles not excluding m. To make the result apply to a continuous body, let dm be any element of mass, V the potential of the whole body at that element, then the mutual potential is

$$W = \tfrac{1}{2}\int Vdm \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

264. The potential of a particle of mass m at distance r from the particle is

$$V = \frac{m}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

It seems that, if we put $r = 0$ in this, we get an infinite value for V. But this arises from our assuming that a finite mass can occupy a point in space, which is clearly absurd. If m is a finite mass it does not occupy a point, but a finite region of space. We can only get the true potential of any body, however small, by regarding it as made up of infinitely small particles and summing the potentials of all the particles.

We shall find the potential of a sphere of radius r at its centre, and show that the limiting value of this potential when $r = 0$ is zero.

Let the sphere be divided into thin concentric shells and let x and $x + dx$ be the radii of the inner and outer surfaces of one of these shells. If ρ is the density, the potential of this shell at its centre is

$$\rho\frac{4\pi x^2dx}{x} = 4\pi\rho xdx \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The potential of the solid sphere is therefore

$$V = 4\pi\rho\int_0^r xdx = 2\pi\rho r^2 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This is a finite quantity for all values of r, and it vanishes when r vanishes.

It follows that the potential of any finite body at a point inside its mass is always finite. For, a sphere with its centre at the point can always be taken which shall extend beyond the boundary of the body, and since the potential of the sphere is finite that of the body must also be finite. Moreover, the potential of an infinitely small body with a finite density is infinitely small. Since the potential of an infinitely small portion dm of a finite body is infinitely small at any point, it is

clear that the potential at the position of dm, due to all the mass except dm, is the same as the potential of the whole, including dm. Hence, in equation (11) of the last article, V is the potential of the whole body at the position of dm.

265. Mutual Potential of a Solid Sphere.—We must first find the potential in a solid sphere at a distance r from the centre.

Let ρ be the density and a the radius. Suppose the sphere divided into thin spherical shells, and let x and $x + dx$ be the internal and external radii of one of these shells.

If the given point is outside the shell of radius x, the potential at that point is the same as if the shell were concentrated into a particle at its centre. This being true for all shells to which the point is external, the potential of the solid sphere of radius r (that is, the sphere whose surface contains the given point) is

$$\frac{\frac{4}{3}\pi\rho r^3}{r} = \frac{4}{3}\pi\rho r^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

When the point lies inside the shell the potential at the point is the same as at the centre. The mass of the shell is $4\pi\rho x^2dx$, and its potential at the centre is $4\pi\rho x dx$. Hence, the potential of the hollow sphere with radii r and a is

$$\int_r^a 4\pi\rho x dx = 2\pi\rho(a^2 - r^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The whole potential at the given point is the sum of the expressions in (1) and (2), namely,

$$V = 2\pi\rho(a^2 - \tfrac{1}{3}r^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

It is worth while to get this potential in another way.

If F is the attraction at a point inside the sphere at distance r from the centre, we know that

$$F = \kappa . \frac{\frac{4}{3}\pi\rho r^3}{r^2} = \kappa . \tfrac{4}{3}\pi\rho r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

But also $$\kappa\frac{dV}{dr} = -F \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Therefore $$V = -\tfrac{2}{3}\pi\rho r^2 + C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Now, the potential at the centre of the sphere, obtained by putting $r = 0$ in (2), is $2\pi\rho a^2$. Hence $C = 2\pi\rho a^2$. Therefore

$$V = -\tfrac{2}{3}\pi\rho r^2 + 2\pi\rho a^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

which agrees with (3).

The mutual potential is

$$W = \tfrac{1}{2}\int_0^a V dm \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The mass at distance r from the centre is the shell $4\pi\rho r^2 dr$. This is the dm in (8). Hence

$$\begin{aligned} W &= \tfrac{1}{2}\int_0^a 2\pi\rho(a^2 - \tfrac{1}{3}r^2)4\pi\rho r^2 dr \\ &= 4\pi^2\rho^2(\tfrac{1}{3}a^5 - \tfrac{1}{15}a^5) \\ &= \tfrac{16}{15}\pi^2\rho^2 a^5 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9) \end{aligned}$$

If M is the total mass of the sphere, this can be written

$$W = \frac{3}{5}\cdot\frac{M^2}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

We can arrive at this result by direct calculation of the work done in bringing the mass from infinity.

Let us suppose that at first the mass was in the form of a uniform spherical shell of infinite radius. The mutual potential of such a shell would be zero, because there is no finite mass in a finite space. Now suppose the sphere is formed gradually by attracting from infinity one thin shell at a time. When the sphere has grown to a radius r, the potential at its surface is $\frac{4}{3}\pi\rho r^2$. The work done in bringing a mass m from infinity to the surface of this sphere will therefore be $\kappa\frac{4}{3}\pi\rho r^2 m$. Hence the work done in bringing up the next shell of mass $4\pi\rho r^2 dr$ is

$$\tfrac{4}{3}\kappa\pi\rho r^2 \,.\, 4\pi\rho r^2 dr = \tfrac{16}{3}\kappa\pi^2\rho^2 r^4 dr \quad . \quad . \quad . \quad . \quad (11)$$

Thus the whole work done in bringing the sphere together is

$$\tfrac{16}{3}\kappa\pi^2\rho^2\int_0^a r^4 dr = \tfrac{16}{15}\kappa\pi^2\rho^2 a^5 \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Since the work is also κW, this gives the same value for W as in (9).

266. Mutual Attraction between Two Parts of the Same Body.—The problem of finding the mutual attraction between two parts of the same body is, in all except a few simple cases, too difficult to be included in this book. The following theorem will be useful in many cases.

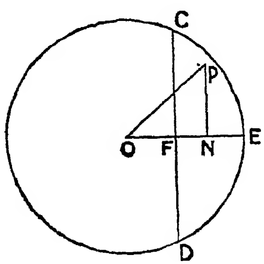


FIG. 142.

The attraction between two parts of the same body is the same as the attraction of the whole on one part.

If the two parts be denoted by A and B, the attraction of the whole on B is the attraction of A on B, together with the attraction of B on itself. But, obviously, the resultant attraction of any body on itself is zero. Hence the theorem follows.

As an example we will find the attraction between two portions of a solid sphere divided by any plane.

Let ρ be the density and a the radius of the sphere. Let CD be the intersection of the dividing plane with the plane of the paper, OE

the radius perpendicular to the dividing plane. Let OF $= b$, ON $= x$, OP $= r$. The attraction at P is along PO, and its magnitude is $\frac{4}{3}\kappa\pi\rho r$; the component of this parallel to EO is $\frac{4}{3}\kappa\pi\rho x$. Thus the component attraction parallel to EO at every point in the plane through N perpendicular to EO is the same. Since the resultant attraction is along EO, the attraction on the whole disc between x and $x + dx$ is

$$\tfrac{4}{3}\kappa\pi\rho x \, . \, \pi\rho(a^2 - x^2)dx = \tfrac{4}{3}\pi^2\kappa\rho^2 x(a^2 - x^2)dx \quad . \quad . \quad (1)$$

Hence the mutual attraction is

$$\tfrac{4}{3}\pi^2\kappa\rho^2\int_b^a x(a^2 - x^2)dx = \tfrac{1}{3}\pi^2\kappa\rho^2(a^2 - b^2)^2 \quad . \quad . \quad . \quad . \quad (2)$$

EXAMPLES ON CHAPTER XI

1. Let AB be a thin uniform rod, and C a point without it. Show that the potential of AB at C depends only on the angles of the triangle ABC and on the density of AB per unit length.

Let O be a point within a triangle ABC, and let DEF be the triangle formed by joining the feet of the perpendiculars dropped from O on the sides of the triangle ABC. Suppose that the sides of the triangles represent uniform rods, all of the same density per unit length. Show that the sum of the potentials of AB, BC, CA, at O, and the sum of the potentials of DE, EF, FD, at O, are equal.

[The potential of AB at C, required in the first part of the question, is

$$\rho \log_e\left(\frac{\operatorname{cosec} A + \cot A}{\operatorname{cosec} B - \cot B}\right)]$$

2. Investigate the attraction of a thin homogeneous circular plate of radius a at a point which is at a perpendicular distance c from the centre of the plate. Remark upon the cases

(i) a infinite, c finite.
(ii) a finite, c infinitesimal.

[In both cases F $= 2\pi\kappa\rho$, as for an infinite plate.]

3. Two thin circular discs with radii a and b, and masses M and m, have a common symmetrical axis, and the distance between their planes is very small compared with the larger radius a. Show that the attraction of either on the other is approximately

$$2\kappa\frac{Mm}{a^2}$$

4. Show that the attraction between a thin circular ring, of radius a and mass M, and a thin rod of mass m and length l lying along the axis of the ring, is

$$\kappa\frac{Mm}{l}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

where r_1 and r_2 are the distances of the ends of the rod from the circumference of the ring.

5. Prove that, if the earth were homogeneous throughout, the decrease in gravitational attraction as one rose through a certain height in a balloon would be approximately twice the decrease as one descended an equal depth in a mine. *London B.Sc.*

6. A thick shell of uniform density is bounded by spherical surfaces which are not concentric: prove that the attraction in the internal cavity is uniform in magnitude and direction. *London B.Sc.*

[If d is the distance between the centres, the attraction is $\frac{4}{3}\kappa\pi\rho d$ parallel to the line joining the centre of the inner, to the centre of the outer, surface.]

7. If the density at distance r from the centre of a sphere of radius a is $\rho\left\{1 - n\left(\frac{r}{a}\right)^2\right\}$, ρ and n being constants, find the attraction and potential at the same distance.

$$\left[\frac{4\pi\kappa\rho r}{3a^2}\left(a^2 - \frac{3n}{5}r^2\right), \text{ and} \right.$$
$$\left.\frac{2\pi\rho}{3a^2}\left\{3a^4\left(1 - \frac{n}{2}\right) - r^2\left(a^2 - \frac{3n}{10}r^2\right)\right\}.\right]$$

8. Find the potential of a solid homogeneous sphere of mass M inside and outside its mass.

If the mass remains unaltered but its density, instead of being uniform, varies as the distance from the centre, show that the potential is unaltered at external points, but that, at an internal point at distance r from the centre, it is diminished by

$$\kappa M(a - r)^2(a + 2r) \div 6a^4$$

London B.Sc.

9. Define the potential of a given mass at a given point. Also define an equipotential surface.

If the attracting mass consists of two equal particles at A and B, separated by a distance a, the surface will be formed by the revolution of a curve about AB: find the equation to this curve.

Certain of the equipotential surfaces will consist of a pair of ovals, one round A and the other round B. Find the condition for oval curves.

[If r and r' are the distances of a point from A and B, one form of the equation to the equipotential surfaces is $\frac{a}{r} + \frac{a}{r'} = n$ (a constant). The condition for a pair of ovals is $n > 4$.]

10. A quantity of matter, attracting according to the law of nature, is uniformly distributed on the circumference of a circle. Prove that the chord of contact of tangents drawn from an external point divides the mass into two parts having equal potentials at the point.

11. If a solid sphere is cut by a plane show that the component, perpendicular to the plane, of the attraction of the whole sphere is the same on every unit mass in the plane section of the sphere.

Prove that the attraction between a portion of a solid sphere bounded by two parallel circular sections of radii y and $(y + dy)$ and the rest of the sphere is $\frac{4}{3}\kappa\pi^2\rho^2 y^3 dy$, ρ being the density of the sphere.

[If x and $(x + dx)$ are the distances of the planes from the centre, note that $ydy = -xdx$.]

12. A uniform solid sphere of mass M and radius a is cut in two by a diametral plane. Show that the resultant attraction between the two halves is

$$\tfrac{3}{16}\kappa M^2 a^{-2}$$

London B.Sc.

[Use the result for the attraction on a disc in the preceding question.]

13. If M denotes the mass and a the radius of a solid homogeneous sphere, prove that the attraction between two portions of it divided by a plane section of radius y is

$$\frac{3}{16} \cdot \frac{\kappa M^2 y^4}{a^6}$$

14. If half the mass of the earth were concentrated in an infinitely thin uniform external crust, the other half being symmetrically distributed throughout the interior, show that at the centre of a relatively small circular gap in the crust the intensity of gravity would be less than its actual value by one-fourth. *London B.Sc., Honours.*

15. Show that a uniform thin spherical shell exerts no attraction on a particle inside the shell.

Show that the attraction at the centre, of any portion cut off by a plane, is proportional to the area of the section. *London B.Sc.*

16. Show that the attraction between the two halves of a spherical shell of surface density σ and radius r is $2\kappa\pi^2\sigma^2 r^2$.

Show also that the attraction between two portions divided by a small circle of radius x is $2\kappa\pi^2\sigma^2 x^2$.

17. A spherical conductor, made up of two hemispheres, is electrified to potential V; show that the two halves repel each other with a force $\frac{1}{4}V^2$. *London Inter. Sci., Honours.*

18. Let AB be a diameter of a spherical shell of uniform surface density. Show how to draw a plane at right angles to AB which will divide the surface into two parts such that their attractions on the mass just surrounding A shall be equal.

[The plane cuts AB at C such that AC = $\frac{1}{4}$AB.]

19. Find the potential at any point of the axis of a thin hemispherical shell. Show from the result that the resultant attraction at points on the axis is always towards the pole of the hemisphere; and verify that the attractions on the two sides differ by $4\pi\kappa\sigma$, where σ denotes the surface density. *London Inter. Arts, Honours.*

[At distance x from the centre of the sphere the potentials are

$$\frac{2\pi\sigma a}{x}\{\sqrt{x^2+a^2} \pm (x-a)\}$$

the upper sign applying to the concave side, and the lower to the convex side of the shell. a denotes the radius.]

20. A uniform spherical shell of radius a and mass σ per unit area is divided into two portions by a circle of radius r. Show that either portion would attract unit mass at the centre of the circular section with a force $2\pi\kappa\sigma\dfrac{a}{a+r}$.

21. If the axis of x be taken along the axis of a disc of radius a and density σ per unit area, show that the force exerted on unit mass at the origin by the disc, whose centre is at x, in the positive direction along the x-axis, is $2\pi\kappa\sigma\left\{\dfrac{x}{\sqrt{(a^2+x^2)}} \mp 1\right\}$, the upper or lower sign being taken according as x is positive or negative. Thence, or otherwise, prove that the attraction of a solid cylinder at a point on its axis inside the cylinder and distant d from the nearer end is

$$2\pi\kappa\rho\{l - 2d - \sqrt{(l-d)^2+r^2} + \sqrt{d^2+r^2}\}$$

l being the length, r the radius, and ρ the density.

22. A right cone of semi-vertical angle α and length h attracts a particle at its vertex. Prove that the force of attraction is proportional to the length of the axis.

Find an expression for the attraction at any point of the axis.

$$\left[2\pi\kappa\rho\left\{h - (d-p)\cos^2\alpha - p\sin^2\alpha\cos\alpha\log\frac{h\sec\alpha + p\cos\alpha + d}{p(1+\cos\alpha)}\right\}\right.$$

for a point at distance p above the vertex. d is the distance of the point from the rim of the base.]

23. If a cone of any form cuts a number of parallel planes, all of which are covered with the same uniform distribution of matter, σ per unit area, show that the attraction, at the vertex of the cone, of the portion of each plane included in the cone, is the same; and prove that the component of this attraction perpendicular to the planes is $\kappa\sigma\omega$ where ω is the solid angle of the cone.

24. Prove that the component, perpendicular to the base, of the attraction at the vertex of a frustum of a cone (of any form) bounded by a plane parallel to the base, is equal to the density, in gravitational units, multiplied by the height of the frustum and the solid angle of the cone.

London Inter. Arts, Honours.

[If the density is expressed in gravitational units the gravitation constant is unity.]

25. Show that the attraction of an indefinitely long thin strip of matter of breadth a and superficial density ρ, at any point P, resolved perpendicular to the plane of the strip, is $2\kappa\rho\theta$, where θ is the angle between the planes through P and the edges of the strip. *London B.Sc., Honours.*

26. Show that the attraction of an infinitely long prism, whose section is an isosceles triangle, at the vertex of a section infinitely distant from both ends, is $2\kappa\rho h\theta$, where h is the height of the section, θ being the angle between the equal sides, and ρ the density of the prism.

27. An infinitely long prism of density ρ has a rectangular section with sides a and b. Show that the attraction at the mid-point of one of the sides of length b of a section infinitely distant from both ends, is

$$4\kappa\rho\left(a\theta - \frac{b}{2}\log_e \sin\theta\right)$$

where $\theta = \tan^{-1}\dfrac{b}{2a}$.

What does this become when b becomes infinite?

[$2\pi\kappa\rho a$.]

28. A hole, bounded by the circle $r = a$, is cut in a uniform lamina of uniform surface density σ whose edge is bounded by the curve $r = be^{\cos\theta}$; prove that the attraction at the origin is $\pi\gamma\sigma$, where γ is the constant of gravitation.

Prove that if the edge of the lamina is $r = f(\theta)$, and the edge of the hole is $r = cf(\theta)$, where c is a constant, the attraction at the origin is zero.

London B.Sc.

29. From a circular plate of radius r_1 a circular hole of radius r_2 is cut, the circles having a common tangent. Show that the attraction of the remainder at the point of contact of the common tangent is $2\kappa\rho\log\dfrac{r_1}{r_2}$, ρ being the mass per unit area.

30. From a uniform triangular plate of height h a triangular piece of height p is cut away by a line parallel to the base. Show that the attraction of the rest at the vertex of the triangle is $2\kappa\rho \sin\alpha \log\frac{h}{p}$ acting along the bisector of the vertical angle, 2α being the magnitude of this angle.

31. Prove that the attraction of a rectangular lamina with sides $2a$ and $2b$, at a point distant x from the lamina and on the line perpendicular to its plane through its middle point, is

$$4\kappa\rho \sin^{-1} \frac{ab}{\sqrt{(x^2 + a^2)(x^2 + b^2)}}$$

32. Show that the work done in collecting the particles of a body from infinity is the integral $\frac{1}{2}\int V dm$ taken throughout the body, V being the potential at the position occupied by dm.

A sphere of radius a has a total mass M, and its density is proportional to the distance from the centre. Prove that the work gained in redistributing the mass (the radius still remaining a) so that the density is uniform is

$$\frac{1}{35}\kappa\frac{M^2}{a}$$

London B.Sc., Honours Maths.

33. Define (i) potential, (ii) tubes of force, and prove the fundamental property of a tube of force.

If the lines of force are circles whose centres are at a fixed point O and whose planes pass through a fixed line OA, compare the intensity of force at two points in free space which lie on the same line of force.

(*London B.Sc., Applied Maths. subsidiary to Honours Physics.*)

[If P and Q are on one line of force

$$\frac{\text{Force at P}}{\text{Force at Q}} = \frac{\sin \text{AOQ}}{\sin \text{AOP}}.\Big]$$

PART II

DYNAMICS OF A PARTICLE

CHAPTER XII

KINEMATICS

267. In accordance with the definition of speed in Chapter I., a particle which has moved through s feet in t seconds along any path whatever, has a speed $\frac{ds}{dt}$. We may call this the velocity of the body if we remember that it is really only the magnitude of the velocity, and that something more is needed to fix the velocity, namely, the direction of the motion.

A velocity, like any other vector, can be resolved into two components parallel to any two lines in its plane, or into three components parallel to any three lines in space. If mutually perpendicular axes OX, OY, OZ, be taken, and the co-ordinates of a moving particle are x, y, z, at any instant the components of its velocity parallel to these three axes are $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$. The magnitude of the resultant velocity is easily seen from a figure to be

$$V = \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right\}} \quad . \quad . \quad . \quad . \quad (1)$$

If the motion is confined to the plane XOY, then $\frac{dz}{dt} = 0$, and therefore

$$V = \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

For a particle moving in three dimensions the component accelerations are $\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}$, and $\frac{d^2z}{dt^2}$. The resultant acceleration is, therefore,

$$\sqrt{\left\{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2 + \left(\frac{d^2z}{dt^2}\right)^2\right\}} \qquad (3)$$

This is not obtained—as might have been expected at first—by differentiating the velocity V with respect to time. For that differential coefficient will only give the rate of increase of the speed, and it was pointed out in Art. 27 that the velocity of a particle may be changing while the speed remains constant. The rate of increase of the speed, that is, $\frac{dV}{dt}$, will only be equal to the acceleration when the particle is travelling in a straight line. In all other cases it will be less than the resultant acceleration.

268. Components of Velocity and Acceleration in any Direction. —Since we shall have very little occasion to refer to motion in three dimensions, we shall suppose that the motion is confined to the xy plane. Let u and v be the components of the velocity of the particle parallel to the axes at time t. Then

$$u = \frac{dx}{dt}, \quad v = \frac{dy}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The component velocity parallel to a line inclined at θ to the axis of x is equal to the sum of the components of u and v parallel to this line. The velocity is, therefore,

$$u \cos \theta + v \sin \theta = \frac{dx}{dt} \cos \theta + \frac{dy}{dt} \sin \theta \quad . \quad . \quad . \quad (2)$$

Similarly, the component acceleration parallel to the same line is

$$\frac{d^2x}{dt^2} \cos \theta + \frac{d^2y}{dt^2} \sin \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If the given line along which the acceleration is required is parallel to the velocity, it is easy to see that

$$\cos \theta = \frac{u}{\sqrt{(u^2 + v^2)}}, \quad \sin \theta = \frac{v}{\sqrt{(u^2 + v^2)}}. \quad . \quad . \quad (4)$$

Hence, the component acceleration along the line of motion is

$$\frac{d^2x}{dt^2} \cdot \frac{u}{\sqrt{(u^2 + v^2)}} + \frac{d^2y}{dt^2} \cdot \frac{v}{\sqrt{(u^2 + v^2)}} \quad . \quad . \quad . \quad . \quad (5)$$

Since $\frac{d^2x}{dt^2} = \frac{du}{dt}$ and $\frac{d^2y}{dt^2} = \frac{dv}{dt}$, this acceleration may be written thus,

$$\frac{1}{\sqrt{(u^2 + v^2)}}\left(u\frac{du}{dt} + v\frac{dv}{dt}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

But since the speed is

$$V = \sqrt{(u^2 + v^2)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

we get on differentiating

$$\frac{dV}{dt} = \frac{1}{\sqrt{(u^2 + v^2)}}\left(u\frac{du}{dt} + v\frac{dv}{dt}\right) \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$\frac{dV_p}{dt} = \frac{1}{[illegible]}\left\{u\frac{dv}{dt} - v\frac{du}{dt}\right\} \ldots (8\text{-}a)$$

which is the same expression as in (6). Hence, the component acceleration along the line of motion is the rate of increase of the speed.

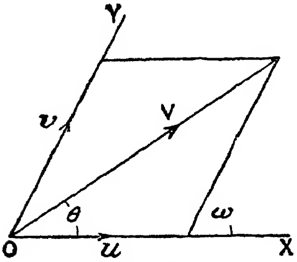


FIG. 143.

The acceleration has generally another component perpendicular to the direction of motion. We shall show what this is a little further on (Art. 272).

269. Inclined Axes.—It will sometimes be convenient to use co-ordinate axes which are not perpendicular to each other. Let the axes OX, OY, be inclined at an angle ω, and let u and v be the component velocities parallel to these axes. From the figure it is clear that

$$V^2 = u^2 + v^2 + 2uv \cos \omega \quad (1)$$

Also, the angle θ, which V makes with OX, is given by

$$\sin \theta = \frac{v}{V} \sin \omega = \frac{v \sin \omega}{\sqrt{(u^2 + v^2 + 2uv \cos \omega)}} \quad . \quad . \quad (2)$$

The component velocities and acceleration are derived from the co-ordinates (or displacements) in exactly the same way as when the axes are at right angles. Thus

$$u = \frac{dx}{dt}, \quad v = \frac{dy}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

And the component accelerations are

$$\left.\begin{aligned} \frac{du}{dt} &= \frac{d^2x}{dt^2} \\ \text{and} \qquad \frac{dv}{dt} &= \frac{d^2y}{dt^2} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

270. Rate of Change of any Vector.—Let $\overrightarrow{OP}$, $\overrightarrow{OQ}$, in Fig. 144, represent a varying vector at times t and $(t + dt)$, the magnitudes at

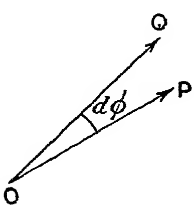


FIG. 144.

these instants being V and $(V + dV)$ respectively, and let the angle between them be $d\phi$, as shown in the figure. The component of the vector $\overrightarrow{OQ}$ along $\overrightarrow{OP}$ is OQ $\cos d\phi$, and consequently the increase of this component in the interval dt is

$$OQ \cos d\phi - OP = (V + dV) \cos d\phi - V$$
$$= dV \cos d\phi - V(1 - \cos d\phi)$$

Therefore the rate of increase of this component is

$$\lim_{dt \to 0} \frac{dV \cos d\phi - V(1 - \cos d\phi)}{dt} = \lim \cos d\phi \frac{dV}{dt} - \lim V \frac{d\phi}{dt} \frac{1 - \cos d\phi}{d\phi}$$
$$= \frac{dV}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

because

$$\frac{1 - \cos d\phi}{d\phi} = \frac{d\phi}{\lfloor 2} - \frac{(d\phi)^3}{\lfloor 4} + $$

which has clearly a limit zero when $d\phi$ is zero, and the limit of $\cos d\phi$ is 1. So the rate of increase of the vector V in its own direction is merely $\frac{dV}{dt}$. This is easy to remember since it is exactly the same as if the vector had no rotation. For example, the acceleration of a particle moving along a straight line with velocity V is $\frac{dV}{dt}$, and what we have just found is that the *component* acceleration in the direction of V is still $\frac{dV}{dt}$ even if the path is curved, and even if the path is not in one plane.

Again, the component of $\overrightarrow{OQ}$ in the direction perpendicular to V and in the plane OPQ is $OQ \sin d\phi$, and, since this component was zero at time t, the rate of increase of this component is

$$\lim_{dt \to 0} \frac{(V + dV) \sin d\phi}{dt} = \lim_{dt \to 0} \frac{V + dV}{V} \cdot \frac{\sin d\phi}{d\phi} \cdot V \frac{d\phi}{dt}$$
$$= V \frac{d\phi}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This component rate of increase of V is in that direction perpendicular to V in which the arrow head of the vector $\overrightarrow{OP}$ is travelling due to the rotation of the vector, the other end of the vector always passing through O. That is, we imagine a plane drawn perpendicular to the vector OP through the point P, and as the vector rotates with the end O fixed, the line of the vector (produced if necessary) cuts this plane in a line, curved or straight. Then the last component is along the tangent to this curved or straight line at P.

If the vector remains in one plane there are only two possible directions perpendicular to V that the component (2) could have. The rule we have just given makes it clear in which of these two directions V is increasing.

Observe that the component rate of increase of V perpendicular to itself is the product of V and the angular velocity of V.

The total rate of increase of V is, of course, the resultant of the two vectors in (1) and (2).

It is useful to remark that all the arguments in this article apply

equally well to a vector which does not pass through a fixed point. The rate of increase of a vector depends only on its successive magnitudes and directions and not on its lines of action.

271. Tangential and Normal Accelerations.—We shall use the result in the last article to find the component accelerations, along and normal to the velocity, of a particle describing a plane curve.

Let V denote the velocity when the particle is at a point P of its path and let ψ denote the angle which V makes with a fixed line OX in the plane of motion. Now we can apply directly the results of the last article. Thus, since the rate of increase of the vector V is its

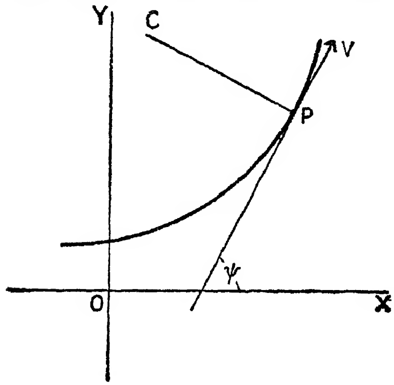


FIG. 145.

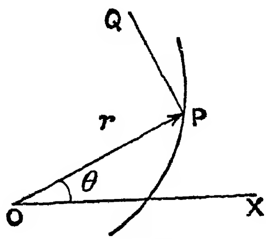


FIG. 145 (*a*).

acceleration, the component accelerations in the direction of V and perpendicular to V are respectively

$$\text{tangential acceleration} = \frac{dV}{dt},$$

$$\text{normal acceleration} = V\frac{d\psi}{dt}.$$

This latter component is in the plane of motion and towards the centre of curvature of the path at P, the point indicated by C in Fig. 145. We can put the latter component in a different, and more usual (but not always more useful) form. Thus suppose ds is the length of arc described by the particle in time dt, so that ds and $d\psi$ are corresponding increments, then

$$\frac{d\psi}{dt} = \frac{d\psi}{ds}\cdot\frac{ds}{dt} = \frac{d\psi}{ds}V = \frac{V}{R}$$

where R is the radius of curvature of the path at the point P. Probably the simplest way of defining radius of curvature is by the expression $\frac{ds}{d\psi}$; but if any other definition be taken it can be proved that it amounts to the same thing. Consequently, the normal acceleration is

$$V\frac{d\psi}{dt} = \frac{V^2}{R},$$

and its direction is along the radius of curvature towards the centre of curvature.

If the path is a circle of radius r then the radius of curvature is r at every point of the path, and the two component accelerations are along the tangent and along the inward-drawn radius.

If the velocity V is constant in magnitude there is still an acceleration provided that the path is not straight, and this acceleration is entirely perpendicular to the velocity.

272. Radial and Transverse Velocities and Accelerations.—Our object here is to express the velocities and accelerations of a particle moving in one plane in terms of its polar co-ordinates r and θ.

Let P be the position of the particle at any instant, and let OP be denoted by r, O being the fixed pole. Then the change of the vector $\overrightarrow{OP}$ in any interval of time is the displacement of the particle in that interval, and consequently the rate of change of $\overrightarrow{OP}$ is the velocity of P. Then, by Art. 270, the components of the velocity along and perpendicular to OP are

$$\text{radial velocity} \quad = \frac{dr}{dt} = u \text{ say,}$$

$$\text{transverse velocity} = r\frac{d\theta}{dt} = v \text{ say,}$$

this latter component being in the direction $\overrightarrow{PQ}$ in Fig. 145 (*a*). To get the components of acceleration we need only find, by the same method, the rate of change of the two vectors u and v. Now the com ponent rates of increase of u are

$$\frac{du}{dt} \text{along } \overrightarrow{OP}, \text{ and } u\frac{d\theta}{dt} \text{ along } \overrightarrow{PQ}.$$

Also the component rates of increase of v are

$$\frac{dv}{dt} \text{along } \overrightarrow{PQ}, \text{ and } v\frac{d\theta}{dt} \text{ along } \overrightarrow{PO}.$$

Thus the total components of acceleration are

$$\frac{du}{dt} - v\frac{d\theta}{dt} \text{ along } \overrightarrow{OP} \text{ and } \frac{dv}{dt} + u\frac{d\theta}{dt} \text{ along } \overrightarrow{PQ}.$$

On substituting for u and v in these we get:

$$\text{radial acceleration} = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2,$$

$$\begin{aligned} \text{transverse acceleration} &= \frac{d}{dt}\left(r\frac{d\theta}{dt}\right) + \frac{dr}{dt}\frac{d\theta}{dt} \\ &= r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} \\ &= \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) \end{aligned}$$

If θ does not vary, that is, if the particle is travelling along the straight line OP, then the transverse acceleration is zero and the

radial acceleration is just $\frac{d^2r}{dt^2}$, which is obviously right for straight-line motion.

Again, if θ is varying and r is constant, that is, for motion in a circle with the pole as centre, then the radial acceleration reduces to $-r\left(\frac{d\theta}{dt}\right)^2$, which is equal to $-\frac{v^2}{r}$. This component acceleration is normal to the path and is precisely the same as the normal component in the last article. Thus we see that the two terms in the radial acceleration are merely the sum of the two terms we should get, (1) by treating θ as constant, and (2) by treating r as constant.

There is no simple or intuitive rule for the transverse acceleration; but we shall later find that it can be expressed easily in terms of moment of momentum.

273. Accelerations in a Rotating Plane.—We shall not deal with the general problem of accelerations of a particle relative to moving axes, but we shall consider one or two simple cases.

Suppose a particle moves in a plane which rotates with a given angular velocity—not necessarily constant—about a fixed line in the plane. We want to express the true accelerations in terms of the accelerations relative to the rotating plane and the angular velocity of the plane.

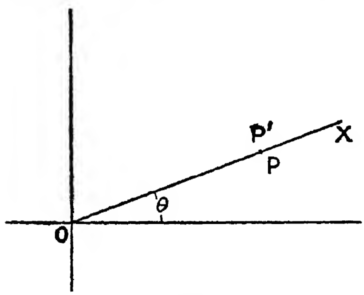


FIG. 146.

Let the fixed axis be taken as y-axis, and let the x-axis be perpendicular to this and in the moving plane. OX is taken in the plane of the paper, and OY is perpendicular to the plane of the paper. Let P denote the particle, and P′ its plan on the plane of the paper. P′ and P are, of course, represented by the same point in the figure. The distance of P above the plane of the paper, that is, the length of PP′, is y; and OP′ $= x$.

The acceleration of P parallel to OY is clearly $\frac{d^2y}{dt^2}$, because this is not affected by the rotation.

The acceleration of P parallel to the plane of the paper is the same thing as the acceleration of its plan P′. Now, by Art. 271, the component acceleration of P′ along OX is

$$\frac{d^2x}{dt^2} - x\left(\frac{d\theta}{dt}\right)^2 = \frac{d^2x}{dt^2} - x\omega^2 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where ω is written for the angular velocity of the plane XOY.

The acceleration of P′ perpendicular to OX and in the plane of the paper is, by the same article,

$$\frac{1}{x}\cdot\frac{d}{dt}\left(x^2\frac{d\theta}{dt}\right) = \frac{1}{x}\cdot\frac{d}{dt}(x^2\omega) \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If the plane were not rotating, the component accelerations parallel to OY and OX would be merely

$$\frac{d^2y}{dt^2} \text{ and } \frac{d^2x}{dt^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and the only difference in these components when the plane rotates is that the second acceleration is diminished by $x\omega^2$, which is the acceleration due to the rotation alone without any other motion. There is, in addition, the acceleration (2) perpendicular to the rotating plane.

274. Axes rotating about an Axis Perpendicular to their Plane. — Suppose the axes OX, OY, to which the position of a moving particle is referred, are rotating about an axis through O perpendicular to their plane with a constant angular velocity ω. To find the true acceleration parallel to the instantaneous positions of the axes.

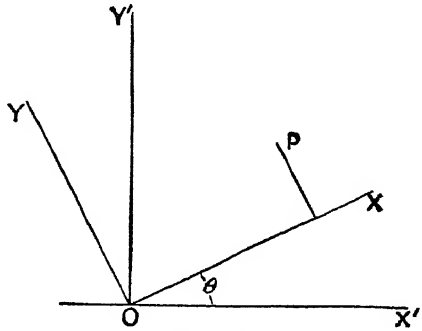


FIG. 147.

Let a pair of perpendicular axes OX′, OY′, be taken fixed in space and in the plane of the rotating axes, and let the co-ordinates of the particle referred to these axes be x', y'. Then

$$y' = y\cos\theta + x\sin\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{dy'}{dt} = \frac{dy}{dt}\cos\theta - y\sin\theta\,.\,\omega + \frac{dx}{dt}\sin\theta + x\cos\theta\,.\,\omega \quad . \quad (2)$$

$$\left.\begin{aligned}\frac{d^2y'}{dt^2} = \frac{d^2y}{dt^2}\cos\theta - 2\frac{dy}{dt}\sin\theta\,.\,\omega - y\cos\theta\,.\,\omega^2 \\ + \frac{d^2x}{dt^2}\sin\theta + 2\frac{dx}{dt}\cos\theta\,.\,\omega - x\sin\theta\,.\,\omega^2\end{aligned}\right\} \quad . \quad . \quad (3)$$

Now $\frac{d^2y'}{dt^2}$ is the true acceleration parallel to OY′. Hence by taking OY′ along OX and OY in turn, we shall get the true accelerations parallel to these lines. Thus putting $\theta = \frac{\pi}{2}$ in (3), we find

$$\text{acceleration along OX} = \frac{d^2x}{dt^2} - x\omega^2 - 2\omega\frac{dy}{dt} = \frac{du}{dt} - x\omega^2 - 2\omega v \quad . \quad (4)$$

and putting $\theta = 0$,

$$\text{acceleration along OY} = \frac{d^2y}{dt^2} - y\omega^2 + 2\omega\frac{dx}{dt} = \frac{dv}{dt} - y\omega^2 + 2\omega u \quad . \quad (5)$$

where u and v are component velocities relative to the rotating axes.

275. Accelerations and Velocities relative to Axes which move in their Plane without rotating.

Let fixed axes O'X', O'Y', be taken parallel to the moving axes OX, OY. Let the co-ordinates of O relative to the fixed axes be ξ, η, and let the co-ordinates of the moving particle relative to the fixed and moving axes respectively be (x', y') and (x, y). Then

$$x' = \xi + x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$y' = \eta + y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Differentiating these twice in succession,

$$\frac{dx'}{dt} = \frac{d\xi}{dt} + \frac{dx}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\frac{dy'}{dt} = \frac{d\eta}{dt} + \frac{dy}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and

$$\frac{d^2x'}{dt^2} = \frac{d^2\xi}{dt^2} + \frac{d^2x}{dt^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\frac{d^2y'}{dt^2} = \frac{d^2\eta}{dt^2} + \frac{d^2y}{dt^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The quantities on the left of equations (3) and (4) are true velocities of the moving particle parallel to the axes, and the quantities on the left of (5) and (6) are true accelerations of the moving particle. Our equations tell us that the velocity of the moving particle is the resultant of the velocity of O relative to the fixed axes and the velocity of the particle relative to the moving axes. Likewise the true acceleration of the moving particle is the resultant of the acceleration of O and the acceleration of the particle relative to the moving axes.

276. As an example of relative motion, suppose a wheel rolls at a uniform rate on a level surface and in one plane. To find the velocity and acceleration of a point on its rim.

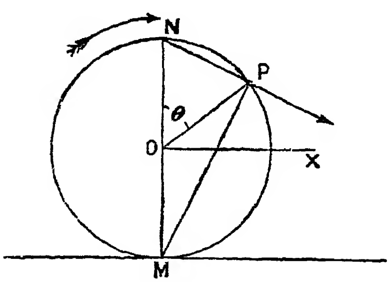


FIG. 148.

Let r be the radius and ω the angular velocity of the wheel. The velocity and acceleration of P are required. Let the line of motion of the centre be taken as the axis of x, and a line through the centre perpendicular to this as the axis of y. The velocity of O is clearly $r\omega$, for in one second the wheel turns through ω radians, and, since there is no sliding, the wheel moves forward $r\omega$. Now the velocity of P relative to the moving axes is also $r\omega$, and its direction is perpendicular to OP. Hence the true velocity of P is the resultant of $r\omega$ in a horizontal direction, and $r\omega$ perpendicular to OP, that is, along the tangent. This

resultant makes equal angles with the horizontal and the tangent at P, and consequently it makes equal angles with the lines MN and OP, which are respectively perpendicular to the horizontal and the tangent at P. Hence the line of the resultant passes through the end N of the vertical diameter. Also, since the tangent at P makes an angle θ with the horizontal, the magnitude of the resultant velocity is

$$2r\omega \,.\, \cos\frac{\theta}{2} = \text{MP} \,.\, \omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The preceding could easily have been proved by making use of the fact that M is the instantaneous centre of rotation of the wheel.

The acceleration of the point P is the acceleration of P relative to the axes through O combined with the acceleration of O. But the acceleration of O is zero. Hence the acceleration of P is just the same as if the wheel rotated without moving forward. Its acceleration is therefore $r\omega^2$ along PO.

277. Acceleration of the Point of Contact of a Circle rolling on a Fixed Circle.

Let O and C be the centres, r and R the radii, of the rolling and fixed circles respectively. Let A be the point of contact of the rolling circle with the fixed circle. OY, OX, are a pair of axes which move with O, but keep their directions always parallel to the original position of OC and the tangent to the circles at A. That is, the directions of the axes remain fixed while their intersection O describes a circle of radius $(R + r)$.

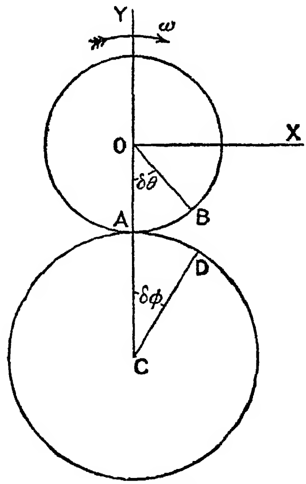


FIG. 149.

The point A describes a circle relative to the moving axes OX, OY, and the point O describes a circle in space. If ω is the angular velocity of the rolling body, the component accelerations of A relative to the moving axes are

$$-r\frac{d\omega}{dt} \text{ parallel to OX} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$r\omega^2 \text{ parallel to OY} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

We have now to find the angular velocity of CO. Suppose that after an interval δt, B comes in contact with D. Then, on account of rolling,

$$r\delta\theta = R\,\delta\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

But when B comes in contact with D, the radius BO will lie along CD. That is, OB will have turned beyond the vertical by an amount $\delta\phi$.

Hence the rolling circle will have turned through $\delta\theta + \delta\phi$ from the first position. But in δt this circle will turn through $\omega\delta t$. Hence

$$\delta\theta + \delta\phi = \omega\delta t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

From (3) and (4) we get

$$(R + r)\delta\phi = r\omega\delta t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Whence

$$\frac{d\phi}{dt} = \frac{r\omega}{R + r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This is the angular velocity of CO. Hence the accelerations of O are

$$(R + r)\frac{d^2\phi}{dt^2} = r\frac{d\omega}{dt} \text{ parallel to OX} \quad . \quad . \quad . \quad (7)$$

and

$$-(R + r)\left(\frac{d\phi}{dt}\right)^2 = -\frac{r^2}{R + r}\omega^2 \text{ parallel to OY} \quad . \quad (8)$$

Adding the accelerations of O and of A relative to the moving axes, we find that the true component accelerations of A are

$$r\frac{d\omega}{dt} - r\frac{d\omega}{dt} = 0 \text{ parallel to OX} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and

$$-\frac{r^2}{R + r}\omega^2 + r\omega^2 = +\frac{Rr}{R + r}\omega^2 \text{ parallel to OY} \quad . \quad . \quad (10)$$

If the fixed and rolling curves are not circles, the results just obtained will be still true, provided we put the radii of curvature of the two curves at the point of contact instead of the radii of the circles For it can be shown that every step in the argument leading to equations (9) and (10) would apply equally well to any two curves if radii of curvature were substituted for the radii of the circles.

278. *A circle of radius* r *rotates about* ~~a~~ its *diameter with constant angular velocity* ω, *and a particle moves at a uniform rate round the circle. To find the acceleration of the particle.*

Let the axis of rotation be taken as axis of y, and a perpendicular diameter as axis of x, just as in Art. 273.

Let n be the angular velocity of the particle in the plane of the circle. If the plane were not rotating, the accelerations parallel to the axes OX, OY would be merely the components of the normal acceleration rn^2 This gives components

$$-xn^2 \text{ and } -yn^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

parallel to OX and OY respectively.

Owing to the rotation the acceleration parallel to OX is diminished by $x\omega^2$. Hence the whole acceleration parallel to OX is

$$-x(n^2 + \omega^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

There is also an acceleration perpendicular to the plane whose magnitude is

$$\frac{1}{x} \cdot \frac{d}{dt}(x^2\omega) = 2\omega\frac{dx}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If t is measured from the instant when the particle was on the x-axis, we have, for positive rotation,

$$\left.\begin{aligned} x &= r\cos nt \\ y &= r\sin nt \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Therefore $$\frac{dx}{dt} = -nr\sin nt = -ny \quad . \quad . \quad . \quad . \quad (5)$$

Hence the three accelerations parallel to OX, OY, and perpendicular to the plane, are respectively,

$$-x(n^2+\omega^2),\ -yn^2, \text{ and } -2yn\omega \quad . \quad . \quad . \quad . \quad (6)$$

279. Acceleration of the end of a string unwinding from a convex curve.

A string is unwound from a convex curve and held tight during the process. To find expressions for the accelerations along and perpendicular to the string.

It is clear that during unwinding the string is always tangent to the curve. If, therefore, P and Q are two neighbouring points on the curve described by the end of the string, the angle $\delta\phi$ between the tangents at P and Q, being equal to the angle between the normals at P and Q, is equal to the angle between the lines of the string at P and Q. Hence, if v is the velocity at P, the tangential acceleration is

$$\frac{dv}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and the normal acceleration is

$$v^2\frac{d\phi}{ds} = v\frac{ds}{dt}\cdot\frac{d\phi}{ds} = v\frac{d\phi}{dt} \quad . \quad . \quad (2)$$

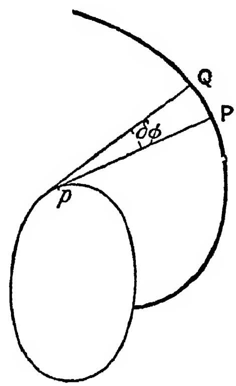


FIG. 150.

If Pp is the tangent at p to the curve on which the string is wrapped, and if the length of Pp is l, the velocity of P is

$$v = l\frac{d\phi}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This makes the expressions for the tangential and normal accelerations become

$$\frac{d}{dt}\left(l\frac{d\phi}{dt}\right) \text{ and } l\left|\frac{d\phi}{dt}\right|^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

respectively.

280. The Hodograph.—If vectors be supposed to be drawn from a fixed point to represent the velocities, at every point of its path, of a particle describing a plane curve, the ends of these vectors will lie on a curve, and this curve is called the *hodograph* of the moving point.

The end of the vector representing the velocity at any point P of

the path of the moving particle is called the point on the hodograph corresponding to P.

The important property of the hodograph is that the velocity of the corresponding point is the acceleration of the particle. This we shall now prove.

Let P and Q be two very near points on the path of a moving particle; $\overrightarrow{Op}$ and $\overrightarrow{Oq}$ represent the velocities at P and Q. The points p and q are therefore on the hodograph, with O as pole. Let δt be the

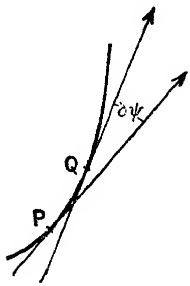


FIG. 151A.

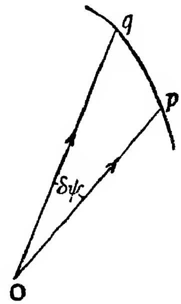


FIG. 151B.

interval of time from P to Q. Now the velocity at Q is greater than the velocity at P by the vector

$$\overrightarrow{Oq} - \overrightarrow{Op} = \overrightarrow{pq}$$

that is, the velocity added in the interval δt is represented in magnitude and direction by $\overrightarrow{pq}$. Since acceleration is increase of velocity per second, the acceleration of the particle is the vector

$$\lim \frac{\overrightarrow{pq}}{\delta t}$$

which is the velocity of the corresponding point in the hodograph. Thus the magnitude and direction of the acceleration of the moving particle are the same as the magnitude and direction of the velocity of the corresponding point in the hodograph, provided, of course, that unit velocity is represented in the hodograph by unit length.

From what we have just proved it follows that the tangential and normal accelerations of the moving particle are the same as the radial and transverse velocities of the corresponding point in the hodograph. Thus the radial velocity in the hodograph is

$$\frac{dv}{dt} \quad \cdots\cdots\cdots\cdots \quad (1)$$

and the transverse velocity in the hodograph is

$$v\frac{d\psi}{dt} = v\frac{d\psi}{ds}\cdot\frac{ds}{dt} = \frac{v^2}{\rho} \quad \ldots\ldots\ldots \quad (2)$$

where ds and ρ refer to the path of the particle.

EXAMPLE.—The hodograph of a particle describing a circle of radius r with constant speed is another circle of radius v. Also the corresponding point in the hodograph moves with uniform speed.

Since the two circles are described in the same time

$$\frac{\text{speed in hodograph}}{v} = \frac{\text{radius of hodograph}}{r} = \frac{v}{r}.$$

Hence the speed in the hodograph is $\frac{v^2}{r}$, and this speed is the normal acceleration of the particle in its path.

281. Hodograph of the motion of the end of a string unwinding from a circle.

A string is unwound from a circle and the free part is kept tight. The hodograph depends, of course, on the rate of unwinding. Let us

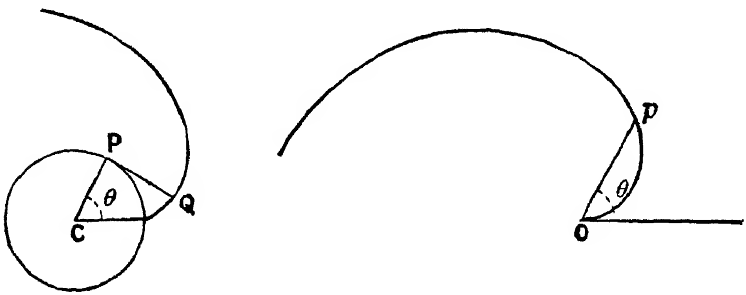


FIG. 152A. FIG. 152B.

suppose, in the first instance, that the straight portion of the string turns with constant angular velocity.

If a is the radius of the circle and ω the angular velocity of the straight piece of string, the velocity of the end when a length $a\theta$ has been unwound is $a\theta\,.\,\omega$. Thus the velocity at Q is PQ$\,.\,\omega$, and the direction of this velocity is perpendicular to PQ, that is, parallel to CP. Hence Op, which represents the velocity at Q, is parallel to CP. The polar equation of the hodograph is therefore

$$r = a\omega\theta,$$

a "spiral of Archimedes."

If, instead of taking the angular velocity of the string constant, we had taken the linear velocity of the end constant, the hodograph would have been a circle with pole O at the centre. Indeed, whatever path a particle describes, if it moves with uniform speed the hodograph of its motion is a circle.

282. From the equations of the path of a particle and the hodograph of its motion to find its acceleration.

The velocity at any point P of the path is given at once by the radius vector of the hodograph which is parallel to the tangent at P. Then the component acceleration of the particle along the inward normal at P is $\frac{v^2}{\rho}$, where ρ is the radius of curvature at P.

Again, if $d\psi$ be the angle between the tangents at P and at a point ds further along the curve, the acceleration along the tangent is

$$\frac{dv}{dt}=\frac{dv}{d\psi}\cdot\frac{d\psi}{ds}\cdot\frac{ds}{dt}=\frac{dv}{d\psi}\cdot\frac{1}{\rho}\cdot v \quad . \quad . \quad . \quad . \quad . \quad (1)$$

But v and ψ are the polar co-ordinates, r and θ, of the point of the hodograph corresponding to P. We may therefore write for the tangential acceleration,

$$\frac{dv}{dt}=\frac{v}{\rho}\cdot\frac{dr}{d\theta}=\frac{r}{\rho}\cdot\frac{dr}{d\theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

ρ being the radius of curvature at the point in the path of the particle corresponding to the point (r, θ) in the hodograph.

The student will probably find it less confusing not to use the symbols r and θ at all, but to call the polar co-ordinates in the hodograph v and ψ.

EXAMPLE.—*Suppose the particle describes the catenary* $s = c \tan \psi$, *and suppose that the equation of the hodograph is* $r = a\sqrt{\sec \theta}$, *or* $v = a\sqrt{\sec \psi}$.

Then the normal acceleration is

$$\frac{v^2}{\rho}=v^2\div\frac{ds}{d\psi}=\frac{a^2 \sec \psi}{c \sec^2 \psi}=\frac{a^2}{c}\cos \psi \quad . \quad . \quad . \quad . \quad (3)$$

The tangential acceleration is

$$\frac{v}{\rho}\cdot\frac{dv}{d\psi}=\frac{1}{2\rho}\cdot\frac{d}{d\psi}(v^2)=\frac{1}{2c \sec^2 \psi}\cdot a^2 \sin \psi \sec^2 \psi=\frac{a^2}{2c}\sin \psi \quad . \quad (4)$$

EXAMPLES ON CHAPTER XII

1. A rod AB moves in any manner whatever. If α and β denote the component velocities (or accelerations) of A and B parallel to any axis, show that the component velocity (or acceleration) of any point P on the rod, parallel to the same axis, is $\frac{b\alpha + a\beta}{a + b}$, where a and b denote the lengths of AP and BP.

2. A rod moves with its ends sliding on rectangular axes OX, OY. If x, y, are the co-ordinates at any instant of a point P on the rod, and if the angular velocity ω of the rod is constant, show that the component accelerations of P are $-x\omega^2$ and $-y\omega^2$. Thence show that the resultant acceleration is OP . ω^2 towards O.

CHAPTER XIII

NEWTON'S LAWS OF FORCE AND RECTILINEAR MOTION

283. Newton's Laws.—In the last chapter we dealt with velocity and acceleration merely as rates of change of geometrical quantities. Velocity is the rate of displacement, and acceleration is the rate of increase of velocity. The theorems proved in the last chapter are therefore purely geometrical ones and have nothing to do with the cause of motion. The main problem in dynamics is to find the motion and displacement of a body caused by given forces, and for this problem we need a connecting link between force and motion. This link is supplied by Newton's second law of force. The science of dynamics is built on an experimental basis which is summed up in Newton's three laws of force. These laws have been given previously in this book, but we shall state them again here.

LAW I.—*Every body continues in a state of rest or of uniform motion in a straight line unless it is compelled by forces to change that state.*

LAW II.—*Change of momentum is proportional to the force producing that change, and is in the direction of the force.*

LAW III.—*To every action there is an equal and opposite reaction.*

284. The first law tells us that, if no forces act on a body, it retains its velocity unaltered in magnitude and direction; or, if it is at rest, it will remain at rest. This law is really implied in the second law.

The momentum of a particle is the product of its mass and its velocity. Momentum must be treated as a vector with the same direction as the velocity. In finding the change of momentum we must find the difference by vector rules. We must take "change of momentum" in the second law with Newton's meaning, namely, "rate of change of momentum." Then the law states that the rate of change of momentum of a body is proportional to the force producing it. To avoid having to define here what the momentum of a finite body is, we shall take a particle as our body.

Thus, if v denotes the velocity of a particle of mass m acted on by a force F, the law states that,

$$\text{rate of change of } mv \propto \mathrm{F};$$

that is,

$$m \times (\text{rate of change of } v) \propto \mathrm{F};$$

or

$$m \times (\text{acceleration}) \propto \mathrm{F} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

It must be remembered that this statement applies to all cases, and does not in the least depend upon the relative directions of the velocity and the force. If the hodograph of the moving particle be drawn, the force at any point of its path is therefore parallel to the tangent at the corresponding point of the hodograph.

If we write f for the acceleration, equation (1) can be written

$$F = Cmf \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where C is some constant. As we pointed out in Chapter I., the most convenient unit of force for dynamics is the one obtained by making $C = 1$. When the units of mass, length, and time, are fixed, equation (2), together with the assumption that $C = 1$, fixes the unit of force also. Then the fundamental equation in dynamics becomes

$$F = mf \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Since the vectors F and mf are equal, their components parallel to any axes will be equal. If then x, y, z, are the co-ordinates of the particle referred to any three axes in space, and X, Y, Z, are the components of F parallel to these axes, we get, on equating the components of F and mf,

$$\left.\begin{aligned} X &= m\frac{d^2x}{dt^2} \\ Y &= m\frac{d^2y}{dt^2} \\ Z &= m\frac{d^2z}{dt^2} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If the motion is confined to the xy plane the last of these equations is unnecessary; and if the motion is along one straight line, taken as x-axis, only the first equation is needed.

285. The statement made in the third law is that when a body A exerts a force on another body B, then B exerts on A a force of equal magnitude in the opposite direction along the same straight line. Either of these forces is called an *action*, and the other is called the corresponding reaction.

This law is strictly true when the forces considered are really action and reaction. It is very common, however, to assume that the actions of two bodies A and B on a third connecting body C are action and reaction. Thus, when a horse drags a barge by means of a rope it is often assumed, and stated, that the reaction of the boat on the horse is the pull of the boat on the rope. But actually there are no direct actions between the horse and the barge. The horse pulls at the rope (or more correctly, at its collar), and the reaction to this pull is the pull of the rope (or its collar) on the horse. The pull of the boat on the rope is the reaction to the pull of the rope on the boat. Whether the backward pull of the boat on the rope is equal and opposite to the forward pull of the horse on the rope depends on the other forces acting on the rope as well as on its mass and acceleration. Generally the directions of the

pulls at the ends of the rope are not even the same. It is true that there is not much error in regarding the actions of the horse and the boat on the rope as equal and opposite, but there is some error, whereas Newton's law is absolutely accurate so far as we can test it experimentally.

There are other cases of forces transmitted from one body to another where the connecting link is not so obvious as in the case of the horse and the boat. Such are electric, magnetic, and gravitation forces. Before we can say with certainty that the action of the earth on the moon, and the action of the moon on the earth, are the action and reaction of Newton's third law, we must be sure that the action is a direct one. If some intervening body or medium transmits the force from one to the other, the actions and reactions are the forces between each body and this medium. The force which accelerates the moon is the action of the medium, and the reaction of the moon is its action on the medium. Newton's third law cannot be taken to mean that the action transmitted through a medium from one body to another is equal and opposite to the action transmitted from the second to the first. Yet this is usually assumed for gravitation forces, and this assumption leads to results which are in accord with observations. Whether this means that the action takes place without the aid of a medium (an idea which, for some strange reason, many scientists of the present day seem to imagine is self-contradictory), or whether there is a medium whose density is infinitely small, and some of whose elastic properties are infinitely great, we have at present no means of deciding.

286. **D'Alembert's Principle.**—If several forces act on a particle of mass m and produce an acceleration f the equation of motion is

$$\Sigma \mathrm{F} = mf \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $\Sigma\mathrm{F}$ denotes the sum of all the forces, obtained, of course, by vector addition.

On transferring mf to the other side, we get

$$\Sigma \mathrm{F} - mf = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This equation tells us that the sum of all the forces minus the vector mf is zero. If we regard $-mf$ as a force, the equation can be interpreted to mean that the sum of all the forces, including $-mf$, is zero; that is, the forces are in equilibrium. Consequently, all the theorems that have been proved concerning a system of forces in equilibrium acting on a particle can be applied to a particle moving with acceleration f if we regard $-mf$ as one of the forces. The quantity mf is called the *effective force* acting on the particle.

The principle just stated is called d'Alembert's principle. It is a very simple deduction from the laws of force, and is a device used to enable us to apply statical rules to dynamical problems when it is convenient to do so. An extremely useful case to which it is applied is that of a particle describing a circle with uniform speed. If ω is the angular velocity of the radius through the particle, and r the radius of the circle it describes, we have shown that it has an acceleration $r\omega^2$

towards the centre of the circle. By d'Alembert's principle the forces acting on the particle, together with the reversed effective force, namely, $mr\omega^2$ acting along the outward-drawn radius, are in equilibrium. In consequence of the frequency with which circular motion is treated as a statical problem by adding the outward force $mr\omega^2$, this quantity has acquired the name "*Centrifugal Force.*" This centrifugal force is a useful fiction, but the student should remember that it is not an actual force on the particle; it is equal to a force invented to make a statical problem with a great resemblance to the dynamical one.

287. Energy Equation.—A dynamical problem is completely solved when the co-ordinates of the body are expressed in terms of the time. When the forces are given, the starting-point is the differential equations of motion in any of their forms. If the forces acting on a particle are expressed in terms of time, we have only to integrate twice each of the equations

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= \mathrm{X} \\ m\frac{d^2y}{dt^2} &= \mathrm{Y} \\ m\frac{d^2z}{dt^2} &= \mathrm{Z} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (1)$$

and the problem will be solved. Generally, however, the forces are known functions of the co-ordinates and not of the time, so that the equations are not solved by simple integrations. But it is usually possible to find one integral of the equations of motion, as we shall now show. The equation is called the *Energy Equation.*

Unless the contrary is stated, it will be assumed that the co-ordinate axes are perpendicular to each other.

Multiplying the first of equations (1) all through by $\frac{dx}{dt}$, we get

$$m\frac{dx}{dt}\cdot\frac{d^2x}{dt^2} = \mathrm{X}\frac{dx}{dt} \quad \ldots\ldots\ldots \quad (2)$$

Let u be written for $\frac{dx}{dt}$ on the left-hand side. Then this equation is

$$mu\frac{du}{dt} = \mathrm{X}\frac{dx}{dt}. \quad \ldots\ldots\ldots \quad (3)$$

Integrating with respect to t,

$$\begin{aligned} \tfrac{1}{2}mu^2 &= \int \mathrm{X}dx + \mathrm{C} \\ &= (\text{work done by force X}) + \mathrm{C} \quad \ldots\ldots \quad (4) \end{aligned}$$

We can get two similar equations from the other two equations in (1). Adding the three results obtained,

$$\tfrac{1}{2}m(u^2 + v^2 + w^2) = \int \mathrm{X}dx + \int \mathrm{Y}dy + \int \mathrm{Z}dz + \mathrm{A} \quad . \quad (5)$$

where $$v = \frac{dy}{dt}, \quad w = \frac{dz}{dt}$$

If U is the resultant velocity, equation (5) can be written

$$\tfrac{1}{2}mU^2 = (\text{work done by all the forces}) + A \quad . \quad . \quad (6)$$

If U_0 is the velocity at the beginning of any displacement, and U the velocity at the end, the equation (6) can be expressed thus:

$$\tfrac{1}{2}mU^2 - \tfrac{1}{2}mU_0^2 = \text{work done during the displacement}. \quad (7)$$

for it is clear that the constant A is $\frac{1}{2}mU_0^2$.

If we know the path by which a particle travels and the force at every point of the path, equation (7) will give U, or rather the increase in $\frac{1}{2}mU^2$. But when the forces have a potential function (Statics, Art. 135), it is not necessary to know the path. For denoting the potential function by V, we have

$$\begin{aligned} -V &= \text{work done from same standard position} \\ &= \int X dx + \int Y dy + \int Z dz \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8) \end{aligned}$$

If V_0 and V are the values of the potential function at the beginning and end of any displacement, then

$$\tfrac{1}{2}mU^2 - \tfrac{1}{2}mU_0^2 = -V + V_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

This is, of course, the equation obtained by integrating three such equations as (3) between limits and adding the results.

288. The quantity $\frac{1}{2}mU^2$ is called the *Kinetic Energy* of the particle m moving with velocity U. It is the equivalent of the work done on the particle from rest. If the particle starts from rest and ends at rest, the total work done on it is zero; that is, equal amounts of positive and negative work have been done on the particle.

The kinetic energy of a particle is its energy due to motion, and this energy can be converted back into work by allowing it to push some other body forward until the particle is brought to rest. The quantity V is the energy due to position. This can also be converted into work by allowing the particle to change its position while it acts on some other body.

In many problems in dynamics it is much easier to start with the energy equation in any of its forms, (6), (7) or (9), than to start with the equations of motion. Besides, the energy equation is better than the equations of motion because it is an integral of those equations, and is therefore one step nearer the solution.

EXAMPLE 1.—The energy equation of a body projected from the earth's surface with a velocity U_0 in any direction is

$$\tfrac{1}{2}mU^2 - \tfrac{1}{2}mU_0^2 = -mgh$$

where h is the height above the surface when the velocity is U. This equation applies only so long as h is small compared with the earth's radius.

EXAMPLE 2.—The attraction between the sun of mass M and a planet of mass m is $\kappa\frac{Mm}{r^2}$ at distance r. The potential function is therefore

$-\kappa\frac{Mm}{r}$. Assuming the sun to be always at rest, and the planet to have a velocity U at distance r, the energy equation is

$$\tfrac{1}{2}mU^2 = \kappa\frac{Mm}{r} + \text{a constant}$$

or
$$\tfrac{1}{2}mU^2 - \kappa\frac{Mm}{r} = \text{a constant.}$$

EXAMPLE 3.—The energy equation of a body under an attractive force kr, r being its distance from the fixed point towards which the force acts is

$$\tfrac{1}{2}mU^2 + \tfrac{1}{2}kr^2 = \text{a constant,}$$

because the work done in the displacement from o to r is $-\tfrac{1}{2}kr^2$. The above energy equation is correct whether the body moves along a straight line or along a curve.

289. Motion in a Straight Line under a Uniform Force.—We will find the velocity and displacement of a particle of mass m moving in a straight line under a constant force F. If x is the displacement of the particle from some fixed point on the line t seconds after some given instant, the equation of motion is

$$m\frac{d^2x}{dt^2} = F \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Therefore
$$u = \frac{dx}{dt} = \frac{F}{m}t + u_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and
$$x = \tfrac{1}{2}\frac{F}{m}t^2 + u_0t + x_0 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

x_0 and u_0 being the values of x and u when $t = 0$.

Let x be measured from the point where the particle was when t was zero. Then $x_0 = 0$, and (3) becomes

$$x = \tfrac{1}{2}\frac{F}{m}t^2 + u_0t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

By eliminating t from (2) and (4), we get

$$u^2 - u_0^2 = 2\frac{F}{m}x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

This could have been obtained at once from the energy equation, which is

$$\tfrac{1}{2}mu^2 - \tfrac{1}{2}mu_0^2 = Fx \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Equation (6) can very easily be verified in this case, for

$$\frac{d^2x}{dt^2} = \frac{du}{dt} = \frac{du}{dx}\,\frac{dx}{dt} = u\frac{du}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Hence, substituting in (1) $u\frac{du}{dx}$ for $\frac{d^2x}{dt^2}$, we get

$$mu\frac{du}{dx} = \mathrm{F} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

from which we get, on integrating with respect to x,

$$\tfrac{1}{2}mu^2 - \tfrac{1}{2}mu_0^2 = \mathrm{F}x$$

290. The results in the last article can be applied to a body falling freely near the earth's surface, and to a body sliding down a smooth inclined plane, if we neglect the resistance of the air. In the former case $\mathrm{F} = mg$, and therefore $\frac{\mathrm{F}}{m} = g$. In the latter case $\frac{\mathrm{F}}{m} = g \sin \alpha$, where α is the inclination of the plane to the horizontal, as we will now show.

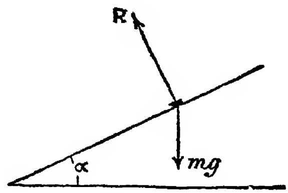


FIG. 153.

The only forces acting on the body are its weight mg and the normal reaction of the plane. Measuring x down the plane, and resolving in this direction,

$$m\frac{d^2x}{dt^2} = mg \sin \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

which is the same equation as (1) in the last article, with $mg \sin \alpha$ for **F**.

When the particle has moved through a distance x down the plane, its vertical descent is $x \sin \alpha$, which we shall call h. Then equation (5) of the last article takes the form

$$u^2 - u_0^2 = 2\frac{g \sin \alpha}{m}x = 2gh \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This could have been obtained directly from the energy equation, for the only force doing work in the motion is the weight, and in a fall of h feet the work done is mgh, whatever be the direction of motion. Hence—

$$\tfrac{1}{2}mu^2 - \tfrac{1}{2}mu_0^2 = mgh \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which agrees with (2).

When a body falls from rest through a vertical distance h, its velocity is given by the equation

$$u^2 = 2gh \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

For this reason the velocity $\sqrt{2gh}$ is called the velocity due to the height (or rather, to the fall) h.

We will consider another case of constant acceleration.

Let CA and CB be two planes inclined at α and β to the horizontal, C being the top of both planes. Masses m_1, m_2, slide on CA, CB respectively, and are connected by a string passing over a frictionless pulley at C.

Suppose μ_1, μ_2, are the coefficients of friction on the planes, and suppose f is the acceleration of m_1 along $\overrightarrow{CA}$ and of m_2 along $\overrightarrow{BC}$. Let T be the tension in the string, R the normal reaction of the plane CA on m_1. Assuming that the velocity of m_1 is in the same direction as f, the friction μ_1R will act along AC.

Resolving perpendicular to CA and along CA for m_1, we get

$$R = m_1 g \cos \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and

$$m_1 f = m_1 g \sin \alpha - \mu_1 R - T$$
$$= m_1 g(\sin \alpha - \mu_1 \cos \alpha) - T . \quad . \quad . \quad (6)$$

The corresponding equation for the particle m_2, the friction on which acts along CB, is

$$-m_2 f = m_2 g(\sin \beta + \mu_2 \cos \beta) - T \quad . \quad . \quad (7)$$

Eliminating T from (6) and (7),

$$(m_1 + m_2)f = m_1 g(\sin \alpha - \mu_1 \cos \alpha) - m_2 g(\sin \beta + \mu_2 \cos \beta) \quad (8)$$

which gives a constant value for f. If this value of f is negative, the particles will finally come to rest. Then they may remain at rest or they may begin to move in the opposite direction. In the latter case the new acceleration in the opposite direction can be obtained by reversing the signs of sin α and sin β in (8). If this new expression for the acceleration is negative also, then clearly no motion is possible, for a body cannot *begin* to move in any direction unless it has a positive acceleration in that direction. We have here, then, a criterion whether the bodies can start from rest or not. If the expression on the right of (8) is negative as it stands, and also when sin α and sin β have their signs changed, then they will remain at rest when once they are at rest. But if either of the two expressions is positive, the motion will begin and continue in the direction of the corresponding acceleration.

The equation will remain equally true if C is the bottom of either of the planes, provided the other plane is on the lower side of that plane, and provided the expression for T, obtained by substituting the value of f in either (6) or (7), never becomes negative.

291. Power is the rate of doing work. The practical unit of power is a horse-power (written H.P.), which is 33,000 foot-lbs. per minute, or 550 foot-lbs. per second, or 550g foot-poundals per second. The theoretical unit of power in dynamics is a foot-poundal per second.

The energy equation for a particle of mass m acted on by a constant force F is

$$\tfrac{1}{2}mu^2 - \tfrac{1}{2}mu_0^2 = Fx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Since the right-hand side is the work done from the point where $x = 0$, we shall get the power on differentiating with respect to t. Thus

$$mu\frac{du}{dt} = F\frac{dx}{dt} = Fu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

We may express this equation thus—

$$\text{rate of increase of kinetic energy} = \text{power} \quad . \quad . \quad . \quad (3)$$

The relation expressed in (3) is a general one for all cases of motion along curved paths and under variable forces. For suppose a particle moves along any curve. Let ds be an element of arc of the curve, v the velocity $\frac{ds}{dt}$, and T the component force acting on the particle along the tangent in the direction in which ds is positive. Then resolving along the tangent,

$$m\frac{dv}{dt} = \mathrm{T} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Multiplying by v throughout,

$$mv\frac{dv}{dt} = \mathrm{T}v = \mathrm{T}\frac{ds}{dt} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

that is,

$$\frac{d}{dt}(\tfrac{1}{2}mv^2) = \mathrm{T}\frac{ds}{dt} = \text{power} \quad . \quad . \quad . \quad . \quad (6)$$

because Tds is the work done in the interval dt corresponding to the displacement ds, and therefore $\mathrm{T}\frac{ds}{dt}$ is the rate at which the forces do work.

If v is constant, the power of all the forces acting on the body is zero. Thus when a railway train has got up full speed the power of the forces on the train is zero. That does not mean that the power of the force exerted by the engine is zero. But the power of the engine's force and the frictional forces is zero. The frictional forces act contrary to the motion, and therefore they do negative work and have negative power. The power of the engine's force is positive and balances the negative power of the frictions, including the air resistance.

292. Simple Harmonic Motion.—A body moves in a straight line so that its acceleration is towards a fixed point O in the line and proportional to its distance from that point. To find the velocity and displacement.

This kind of motion is called *simple harmonic* motion.

Let x be the displacement from the fixed point O, and let the acceleration be kx towards O. The acceleration in the direction in which x increases, that is, away from O when x is positive, is $\frac{d^2x}{dt^2}$. Hence

$$\frac{d^2x}{dt^2} = -kx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This equation is correct whether x is positive or negative. For when x is negative the right-hand side is positive, which means that the acceleration $\frac{d^2x}{dt^2}$ is in the direction of increasing x, and this is right because on the negative side of O the acceleration is towards O and consequently positive.

When the acceleration is a function of x as in (1), we must either

use the energy equation as the starting-point, or obtain the equivalent equation by writing

$$u\frac{du}{dx} \text{ for } \frac{d^2x}{dt^2}$$

Then (1) becomes $$u\frac{du}{dx} = -kx \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Integrating with respect to x,

$$\tfrac{1}{2}u^2 = -\tfrac{1}{2}kx^2 + A \quad . \quad . \quad . \quad . \quad (3)$$

Suppose $u = 0$ when $x = a$. Putting these values in (2),

$$0 = -\tfrac{1}{2}ka^2 + A$$

This gives A in terms of a. Now (3) can be written

$$\tfrac{1}{2}u^2 = \tfrac{1}{2}k(a^2 - x^2) \quad . \quad . \quad . \quad . \quad (4)$$

Whence $$\frac{dx}{dt} = \sqrt{k}\sqrt{(a^2 - x^2)} \quad . \quad . \quad . \quad . \quad (5)$$

and therefore $$\sqrt{k}\frac{dt}{dx} = \frac{1}{\sqrt{(a^2 - x^2)}} \quad . \quad . \quad . \quad . \quad (6)$$

Integrating with respect to x,

$$\sqrt{k}t + \beta = \sin^{-1}\frac{x}{a} \quad . \quad . \quad . \quad . \quad (7)$$

or $$x = a\sin(\sqrt{k}t + \beta) \quad . \quad . \quad . \quad . \quad (8)$$

This shows that the particle oscillates between the two points $x = a$ and $x = -a$. The quantity a is called the *amplitude* of the motion, and the angle β is called the *phase angle.*

A complete oscillation is a displacement from one end of the path to the other end and back again. When the particle is at the end $x = a$ of its path, we have

$$a = a\sin(\sqrt{k}t + \beta) \quad . \quad . \quad . \quad . \quad (9)$$

Therefore $$\sqrt{k}t + \beta = \frac{\pi}{2} + n \cdot 2\pi \quad . \quad . \quad . \quad . \quad (10)$$

where n is any integer. From (10) we get

$$t = \frac{\pi}{2\sqrt{k}} - \frac{\beta}{\sqrt{k}} + \frac{2\pi}{\sqrt{k}} \times (0 \text{ or } 1 \text{ or } 2 \text{ or } 3 \text{ or } 4, \text{ etc.}) \ . \ (11)$$

The difference τ between two successive values of t given by (11), that is, the time taken by the particle to make a complete oscillation, is

$$\tau = \frac{2\pi}{\sqrt{\kappa}} \quad . \quad . \quad . \quad . \quad . \quad (12)$$

This does not depend on the amplitude a. Whether the particle makes large or small oscillations, provided (1) is the accurate equation

of motion, the time of oscillation depends only on k. In future, as soon as the student obtains an equation of motion of the type

$$b\frac{d^2\theta}{dt^2} = -c\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

he should know that it denotes oscillatory motion with a period

$$2\pi\sqrt{\frac{b}{c}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

provided $\frac{b}{c}$ is positive. If $\frac{b}{c}$ is negative the motion is not oscillatory.

The velocity, obtained by differentiating (8), is

$$\frac{dx}{dt} = \sqrt{k}a \cos(\sqrt{k}t + \beta) \quad . \quad . \quad . \quad . \quad . \quad (15)$$

This gives the velocity in terms of t, and (4) gives the velocity in terms of x.

293. Other Forms of the Result for Simple Harmonic Motion.— On integrating equation (6) of the last article we were quite at liberty to take $\cos^{-1}\frac{x}{a}$ instead of $\sin^{-1}\frac{x}{a}$. Then the equation corresponding to (8) would have been

$$x = a\cos(\sqrt{k}t + \beta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

That this is a solution of the original differential equation is easily seen on differentiating twice and substituting in the equation. Thus from (1)

$$\frac{d^2x}{dt^2} = -ka\cos(\sqrt{k}t + \beta)$$
$$= -kx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which is the differential equation (1) of the last article whose solution was required.

Again, taking the solution obtained in the last article and expanding $\sin(\sqrt{k}t + \beta)$, we have

$$x = a\sin(\sqrt{k}t + \beta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$
$$= a\cos\beta\sin\sqrt{k}t + a\sin\beta\cos\sqrt{k}t$$

Now we may write C for $a\cos\beta$ and D for $a\sin\beta$, thus replacing two constants by two other constants. Then we get

$$x = \mathrm{C}\sin\sqrt{k}t + \mathrm{D}\cos\sqrt{k}t \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This form of solution is sometimes more convenient to use than those in (1) or (3) of this article, but all three solutions are equivalent, and any one form can be changed into any other.

In the same way as we have shown that (1) is a solution of the differential equation

$$\frac{d^2x}{dt^2} = -kx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

we may show that (4) is also a solution. And since (4) contains two independent constants it must be the general solution. For the general solution of a differential equation of the second order is any solution of the equation which contains two independent constants. Now (4) is a solution of (5), as can be shown by differentiating, and it contains two independent constants; therefore it is the general solution.

294.—There is a simple geometrical way of representing simple harmonic motion. Suppose the equation for the displacement is

$$x = a \cos (ct + \beta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let a circle of radius a be described with centre O cutting the line of motion in A and A′. Let M be the position of the particle at time t, and let P be a point on the circle such that PM is perpendicular to AA′, and let the angle AOP $= \theta$. Then

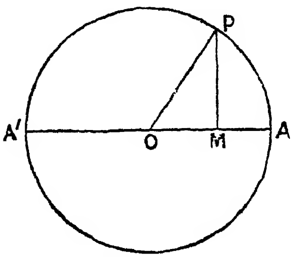


FIG. 154.

$$\text{OM} = x = a \cos (ct + \beta) \quad (2)$$

But also $\text{OM} = a \cos \theta \quad . \quad . \quad . \quad . \quad (3)$

Hence $\quad \theta = ct + \beta + 2n\pi \quad . \quad (4)$

where n is some integer.

Differentiating (4) with respect to t,

$$\frac{d\theta}{dt} = c \quad . \quad . \quad . \quad (5)$$

Thus OP rotates with constant angular velocity c radians per second, and M is the projection of P on the line of motion.

295. When a particle oscillates about a position of stable equilibrium the motion is nearly always approximately simple harmonic. Assuming Hooke's law to be correct, a body oscillates at the lower end of an elastic string with exactly simple harmonic motion. In all cases when a body oscillates under elastic forces the motion is approximately simple harmonic.

EXAMPLE 1.—*A particle is attached to the middle of a uniform elastic string which is stretched between two points* A *and* B *on a smooth horizontal table. If the particle is pulled to some point in the line* AB *and then let go, to find the subsequent motion, assuming that both strings were tight when the particle was let go.*

Let AB $= 2a$, and let $2l$ denote the unstretched length of the string. Let x be the displacement of the particle from O, the mid-point of AB, at any instant. Let the tension in each half of the string be the product of k and the extension, and let m be the mass of the particle.

The extensions of the two halves of the string are $(a - x - l)$ and $(a + x - l)$; the tensions are therefore $k(a - x - l)$ and $k(a + x - l)$. The force acting on the particle is therefore $2kx$ towards O, the difference of the tensions. Hence the equation of motion is

$$m\frac{d^2x}{dt^2} = -2kx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The motion is therefore simple harmonic with a period

$$\tau = 2\pi\sqrt{\frac{m}{2k}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

EXAMPLE 2.—*A particle is attached to the mid-point of a vertical string. The particle oscillates horizontally while the ends are prevented from moving horizontally and the tension kept constant. To find the time of a small oscillation.*

Let A and B be the ends of the string, and P the position of the particle, T is the tension in the string, and x the horizontal displacement of P.

The force acting on P is

$$2T\frac{OP}{AP} = 4T\frac{x}{l} \text{ nearly} \quad . \quad . \quad . \quad . \quad (1)$$

where l is the length of the string.

The equation of motion is therefore

$$m\frac{d^2x}{dt^2} = -4\frac{T}{l}x \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence the motion is simple harmonic, and the period is

$$\tau = 2\pi\sqrt{\frac{ml}{4T}} = \pi\sqrt{\frac{ml}{T}} \quad . \quad . \quad . \quad (3)$$

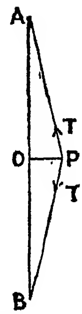


FIG. 155.

The conditions assumed in this example are very nearly realised when a string is tied to the two points A and B. The tension is, however, slightly increased when P is displaced from O. The best kind of string is one which stretches considerably under tension, for then a slight increase in length will scarcely affect the tension.

The conditions are also nearly realised by fastening one end of an inextensible string at A, and attaching a weight at the lower end, the string passing through a small ring at B so as to prevent horizontal motion. The weight is attached, of course, below B. The mass attached must be much greater than m.

296. A particle suspended at the end of an elastic string or spiral spring.

Let x be the extension of the string at any instant, and kx the corresponding tension. There are now two forces acting on the particle in the line of motion, namely, the tension of the string and the weight of the particle.

The equation of motion is

$$m\frac{d^2x}{dt^2} = mg - kx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Therefore
$$\frac{d^2x}{dt^2} = -\frac{k}{m}\left(x - \frac{mg}{k}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Now putting
$$z = x - \frac{mg}{k} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

we get
$$\frac{d^2z}{dt^2} = \frac{d^2x}{dt^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and therefore equation (2) becomes

$$\frac{d^2z}{dt^2} = -\frac{k}{m}z \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

This gives simple harmonic oscillations with period

$$\tau = 2\pi\sqrt{\frac{m}{k}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

On examining (4) it is obvious that z is the downward displacement of the particle measured from the point where $x = \frac{mg}{k}$, that is, from the point where kx, the tension, is equal to the weight. But this is the position of equilibrium of the particle. Hence the particle executes simple harmonic oscillations about the equilibrium position.

Equation (1) is the correct equation of motion of the particle only so long as the string remains stretched. If the oscillations are so large that the string is sometimes slack, the correct form of the equation of motion while the string is slack is

$$m\frac{d^2x}{dt^2} = mg \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

since the string has no influence on the motion provided its mass is negligible, as we have assumed in the problem.

It follows from the preceding that if the equation of motion of a particle is of the form

$$\frac{d^2x}{dt^2} = -c^2x + f \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

the motion has exactly the same character as if the constant f were absent, but the mean position of the particle is displaced, in the positive direction, a distance $\frac{f}{c^2}$ from the position it would occupy if f were zero. The solution of the equation is

$$x = \frac{f}{c^2} + \mathrm{A}\cos ct + \mathrm{B}\sin ct \quad . \quad . \quad . \quad . \quad . \quad (9)$$

the constants A and B being determined by the initial conditions.

297. A Numerical Example.—*When a mass weighing* 4 *ounces is at rest supported by a certain elastic string, the extension of the string is* 4 *inches. If the mass is raised* 2 *inches from its equilibrium position and then let go, to find the time of oscillation and the displacement* t *seconds after the start.*

Taking the tension of the string to be given by

$$T = kx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

we find that, since $T = \frac{g}{4}$ poundals when $x = \frac{1}{3}$ foot,

$$k = \tfrac{3}{4}g \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The equation of motion is therefore

$$\tfrac{1}{4}\frac{d^2x}{dt^2} = \tfrac{1}{4}g - \tfrac{3}{4}gx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Consequently $$\frac{d^2x}{dt^2} = g - 3gx = -3g(x - \tfrac{1}{3}) \quad . \quad . \quad . \quad . \quad (4)$$

The solution of this equation is

$$x = \tfrac{1}{3} + A\cos\sqrt{3g}t + B\sin\sqrt{3g}t \quad . \quad . \quad . \quad (5)$$

Now by the conditions of the problem

$$\left.\begin{array}{l} x = +\tfrac{1}{6} \\ \dfrac{dx}{dt} = 0 \end{array}\right\} \text{when } t = 0 \quad . \quad . \quad . \quad . \quad . \quad (6)$$

But from (5)

$$\frac{dx}{dt} = \sqrt{3g}\{-A\sin\sqrt{3g}t + B\cos\sqrt{3g}t\} \quad . \quad . \quad . \quad (7)$$

To make this zero when $t = 0$ we must have $B = 0$. Then the first of the conditions (6) gives

$$\tfrac{1}{6} = \tfrac{1}{3} + A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Hence $$A = -\tfrac{1}{6}$$

Thus finally $$x = \tfrac{1}{3} - \tfrac{1}{6}\cos\sqrt{3g}t \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and the time of oscillation is $\frac{2\pi}{\sqrt{3g}}$, which is about 0·62 of a second.

In equation (9) x is, of course, expressed in feet.

298. Attraction varying inversely as the Square of the Distance from a Fixed Point.—Since the attraction varies inversely as the square of the distance, the acceleration towards the fixed point varies in the same way. Hence the differential equation of motion takes the form

$$\frac{d^2x}{dt^2} = -\frac{k}{x^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

when x is positive.

Writing u for $\frac{dx}{dt}$ and $u\frac{du}{dx}$ for $\frac{d^2x}{dt^2}$ in (1), we get

$$u\frac{du}{dx} = -\frac{k}{x^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

Integrating $$\tfrac{1}{2}u^2 = \frac{k}{x} + G \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

Now suppose the body is falling inwards, that is, towards the attracting centre, and suppose a was the value of x when u was zero. With this condition (3) becomes

$$\tfrac{1}{2}u^2 = k\left(\frac{1}{x} - \frac{1}{a}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

Therefore

$$\frac{dx}{dt} = -\sqrt{2k}\sqrt{\left(\frac{1}{x} - \frac{1}{a}\right)} = -\sqrt{2k}\cdot\frac{\sqrt{a-x}}{\sqrt{ax}} \quad . \quad . \qquad (5)$$

The negative value of the root is taken because we have assumed $\frac{dx}{dt}$ to be negative in supposing that the particle was falling *towards* the centre of attraction.

From (5) we get

$$\sqrt{\frac{2k}{a}}\cdot\frac{dt}{dx} = -\sqrt{\frac{x}{a-x}}$$

Therefore $$\sqrt{\frac{2k}{a}}t + C = -\int\sqrt{\frac{x}{a-x}}dx$$

Put $$x = a\cos^2\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

Then $$dx = -2a\sin\theta\cos\theta d\theta$$

and $$\sqrt{\frac{x}{a-x}} = \frac{\cos\theta}{\sin\theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

Therefore $$\sqrt{\frac{2k}{a}}t + C = a\int 2\cos^2\theta d\theta$$

$$= a\int(1 + \cos 2\theta)d\theta$$

$$= a(\theta + \tfrac{1}{2}\sin 2\theta) \quad . \quad . \quad . \quad . \qquad (8)$$

As θ varies from 0 to $\frac{\pi}{2}$, the distance x varies from a to 0. We are at liberty to let θ vary from 0 to $-\frac{\pi}{2}$ to get the same variation of x, but then we should have to take a negative sign on the right-hand side of (7) so as to make that side give the positive root of the left. In this case the signs would be reversed on the right of (8); but since θ would now be a negative angle the result would be the same as before.

If t is measured from the instant when $x = a$, then the constant C must be zero, since θ and t have to vanish simultaneously. The time t_1 taken by the body to fall from $x = a$ to $x = 0$ is therefore given by

$$\sqrt{\frac{2k}{a}}t_1 = a\left(\frac{\pi}{2} + \frac{1}{2}\sin\pi\right)$$

$$= \frac{\pi a}{2}$$

Hence $$t_1 = \frac{\pi a^{\frac{3}{2}}}{2\sqrt{2k}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

For outward motion the constant G in (3) may be either positive or negative. Consequently we may write

$$\left(\frac{dx}{dt}\right)^2 = 2k\left(\frac{1}{x} \pm \frac{1}{a}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Then, since the outward velocity is positive,

$$\frac{dx}{dt} = +\sqrt{\frac{2k}{a}}\cdot\sqrt{\frac{a \pm x}{x}}$$

and therefore $$\sqrt{\frac{2k}{a}}t + C = \int\sqrt{\frac{x}{a \pm x}}dx$$

Now take $x = a\sin^2\theta$, or $x = a\sinh^2 z$, according as the lower or the upper sign is correct for the motion we are dealing with.

Taking the negative sign,

$$\sqrt{\frac{2k}{a}}t + C = a\int 2\sin^2\theta d\theta$$
$$= a\int(1 - \cos 2\theta)d\theta$$
$$= a(\theta - \tfrac{1}{2}\sin 2\theta) \quad . \quad (11)$$

If we take the upper sign we get

$$\sqrt{\frac{2k}{a}}t + C = a\int 2\sinh^2 z\, dz$$
$$= a\int(\cosh 2z - 1)dz$$
$$= a(\tfrac{1}{2}\sinh 2z - z)$$
$$= a(\sinh z\cosh z - z)$$
$$= \sqrt{x(x+a)} - a\log_e\frac{\sqrt{x} + \sqrt{x+a}}{\sqrt{a}} \quad . \quad (12)$$

The constant C will be zero if $x = 0$ when $t = 0$. Equation (10) shows that the lower sign applies to the case where the body comes to rest at a distance a from the centre of attraction. After that the body begins to fall inwards. But the positive sign applies to the case where

the velocity is never zero and the velocity at infinity is finite. In the first case the maximum value of θ is $\frac{\pi}{2}$, and therefore the maximum value of the right-hand side of (11) is $\frac{1}{2}a\pi$. Since C will be zero in this equation if t denotes the number of seconds since x was zero, it follows that the time taken in travelling from the centre of force to rest at distance a is

$$t_2 = \frac{\pi a^{\frac{3}{2}}}{2\sqrt{2k}}$$

which is the same as t_1 in (9), the time for the reversed motion, as we should expect. In equation (12), however, there is no limit to x, and therefore no limit to t.

299. If the body has fallen inwards from a very great distance, that is, if a is very large compared with x in equation (4) of the last article, we may neglect $\frac{1}{a}$. Then

$$\frac{dx}{dt} = -\sqrt{2k}\frac{1}{\sqrt{x}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Therefore $$x^{\frac{1}{2}}\frac{dx}{dt} = -\sqrt{2k}$$

and consequently $$\tfrac{2}{3}x^{\frac{3}{2}} = -\sqrt{2k}t + C$$

The velocity $\sqrt{\frac{2k}{x}}$ is called the velocity due to a fall from infinity. If a body were attracted by the sun from a very great distance, and its velocity were zero (or very small) at that distance, then its velocity at distance x from the sun's centre would be the velocity given in (1). This is, of course, only true on the assumption that the sun is the only attracting body. Actually the planets would also have some influence on the velocity.

If the motion is away from, instead of towards, the attracting centre, the positive value of the root must be taken in (1). In this case the body has not fallen from rest at infinity, but it is projected outwards with a velocity which will carry it to infinity and leave it at rest there. To put it in another way, we may say that a body projected from the attracting centre with a velocity greater than $\sqrt{\frac{2k}{x}}$ will never return, for its velocity will carry it to an infinite distance; and a body projected with a velocity less than the above will come to rest at some finite distance, and, neglecting the attraction of other bodies, the projected body will return to the body from which it was projected.

300 **Bodies projected from the Surface of the Earth.**—The earth's attraction on a body of mass m at a distance x from its centre is $mg\frac{r^2}{x^2}$, where r is the radius of the earth. Its acceleration is

thus $g\dfrac{r^2}{x^2}$. The velocity at x which would be just sufficient to carry the body to infinity, that is, out of the earth's influence, is

$$u = \sqrt{\frac{2gr^2}{x}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Putting r for x in this, we get the velocity with which a body must be projected from the earth's surface so that it shall never return. This velocity is

$$u = \sqrt{2gr} = \sqrt{\frac{2 \times 32 \times 3960}{1760 \times 3}} \text{ miles per sec.}$$

$$= 6{\cdot}93 \text{ miles per sec.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Thus a body projected from the earth's surface with a velocity of 7 miles per second will never return to the earth. But this velocity would not carry the body out of the solar system, because the sun's attraction would be too great. We shall now calculate the velocity necessary to carry a body from the earth's surface out of the solar system.

301. We will find the velocity with which a body must be projected from the earth's surface to carry it out of the influence of the earth's and sun's attractions, assuming the earth to be at rest. This will not, it is true, be the same as when we allow for the earth's orbital motion, but it will not be much different if the direction of projection is perpendicular to the earth's direction of motion.

If x and R denote the distances of the body from the earth's and sun's centres, its potential energy is

$$-\kappa\left(\frac{mE}{x} + \frac{mS}{R}\right)$$

where E and S denote the earth's and sun's masses, m the mass of the body, and κ the constant of gravitation.

The energy equation is therefore

$$\tfrac{1}{2}u^2 - \kappa\left(\frac{E}{x} + \frac{S}{R}\right) = \text{a constant}$$

To make the velocity zero when x and R are infinite the constant must be zero. Thus, if r denotes the radius of the earth, the velocity at the earth's surface which will carry a body to infinity is given by

$$u^2 = 2\kappa\left(\frac{E}{r} + \frac{S}{R}\right)$$

$$= 2\kappa\frac{E}{r}\left(1 + \frac{Sr}{ER}\right)$$

Now $S = 328{,}000E$ approximately, and $R = 92$ million miles, $r = 4000$ miles. Consequently

$$\frac{Sr}{ER} = 14{\cdot}3$$

Therefore $u = \sqrt{2\kappa\frac{E}{r}}(1 + 14\cdot3)^{\frac{1}{2}}$

$= 6\cdot93 \times \sqrt{15\cdot3}$ miles per sec., by (2) of the last article,

$= 27$ miles per sec. nearly.

302. We will work a numerical example to illustrate the formulæ giving the time for the inverse square law of attraction.

Two similar spheres, each of mass 2000 *lbs. and radius one foot, are placed with their centres* 6 *feet apart. If they are acted on by no forces but their mutual gravitational attractions, how long will they be before coming into contact?*

Since there is the same force acting on each sphere, they will meet halfway. Let x be the distance of each sphere from the fixed point which is always midway between them. Then $2x$ is their distance apart, and the force acting on each is $\frac{\kappa m^2}{(2x)^2}$, where κ is the constant of gravitation and m the mass of one sphere.

The equation of motion of either sphere is

$$m\frac{d^2x}{dt^2} = -\frac{\kappa m^2}{4x^2}$$

or

$$\frac{d^2x}{dt^2} = -\frac{\kappa m}{4}\cdot\frac{1}{x^2}$$

To get the relation between x and t we have to put 0 for C, 3 for a, and $\frac{\kappa m}{4}$ for k, in equations (6) and (8) of Art. 298. Then

$$\sqrt{\frac{2\kappa m}{4\cdot3}}t = 3(\theta + \tfrac{1}{2}\sin 2\theta)$$

When the spheres come in contact $x = 1$, and therefore $\theta = 0\cdot9553$, $3\sin 2\theta = 2\sqrt{2}$. Consequently

$$\sqrt{\frac{\kappa m}{6}}t = 3(0\cdot9553) + \sqrt{2}$$

$$= 4\cdot280$$

By equation (2), Art. 239, the value of κ when force is measured in poundals is $\frac{1}{9\cdot4 \times 10^8}$. Therefore

$$t = 4\cdot28\sqrt{\frac{6 \times 9\cdot4 \times 10^8}{2000}} \text{ seconds}$$

$$= 2 \text{ hours nearly}$$

CHAPTER XIV

MOTION IN TWO DIMENSIONS

303. The equations of motion of a particle of mass m at the point whose co-ordinates (x, y), referred to any axes in the plane of its motion, are

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= \mathrm{X} \\ m\frac{d^2y}{dt^2} &= \mathrm{Y} \end{aligned}\right\} \quad \ldots \quad (1)$$

The axes need not be at right angles to each other. The equations (1) are true for oblique axes provided the forces X and Y are the components parallel to these axes.

In order to find the path described by the particle x and y must be found from equations (1). If X and Y are given functions of t, then x and y will be found in terms of t, and the equation of the path will be obtained by eliminating t from the equations for x and y.

304. **Motion under Gravity.**—Let the axis OY be taken vertically and downwards, and the axis OX horizontal. The equations of motion are then

$$\frac{d^2y}{dt^2} = g \quad \ldots \quad (1)$$

$$\frac{d^2x}{dt^2} = 0 \quad \ldots \quad (2)$$

Let u and v denote the horizontal and vertical components of the velocity. Then

$$\frac{dy}{dt} = v = gt + v_0 \quad \ldots \quad (3)$$

$$\frac{dx}{dt} = u = \text{constant} \quad \ldots \quad (4)$$

v_0 denoting the value of v when $t = 0$.

Integrating these equations again,

$$y = \tfrac{1}{2}gt^2 + v_0t + y_0 \quad \ldots \quad (5)$$

$$x = ut + x_0 \quad \ldots \quad (6)$$

If the origin be taken at the position of the particle when $t = 0$, then $x_0 = 0, y_0 = 0$. Hence

$$y = \tfrac{1}{2}gt^2 + v_0 t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$x = ut \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Eliminating t from these equations, we get, as the equation of the path,

$$y = \frac{g}{2u^2}x^2 + \frac{v_0}{u}x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

This can be written

$$\frac{2u^2}{g}\left(y + \frac{v_0^2}{2g}\right) = \left(x + \frac{uv_0}{g}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Now the equation $\qquad 4a(y + \beta) = (x + \gamma)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$

is the equation of a parabola with vertical axis and concavity downwards, and vertex at $y = -\beta$, $x = -\gamma$. Also the latus rectum is $4a$. Hence the equation (10) gives a parabola whose latus rectum is $\frac{2u^2}{g}$ and whose vertex is at $\left(-\frac{uv_0}{g}, -\frac{v_0^2}{2g}\right)$.

305. If the particle is projected above the horizontal, v_0 will be a negative quantity. If we denote $-v_0$ by v', then v' is the *upward* component of the velocity of projection. Then the co-ordinates of the vertex of the parabola are $\left(\frac{uv'}{g}, \frac{-v'^2}{2g}\right)$. Since the vertex of the parabola is the highest point reached by the particle, this highest point is at a height $\frac{v'^2}{2g}$ above the level of the point of projection.' This result can, of course, be obtained by ignoring altogether the horizontal motion and assuming that the particle is projected vertically upwards with velocity v'.

To find where the particle meets the x-axis on its downward motion, put $y = 0$ in (9). This gives

$$x = 0, \quad \text{or} \quad x = -\frac{2uv_0}{g} = +\frac{2uv'}{g} \quad . \quad . \quad . \quad . \quad (12)$$

This second result is called the *range* on the horizontal plane through the point of projection. The solution $x = 0$ merely means that the particle starts from the origin.

If v_0 is positive, that is, if the particle is projected below the horizontal, the range on the horizontal plane through O is negative (supposing u positive). But the particle never does meet this horizontal plane after projection in this case. But supposing the path continued backwards, the parabola of which (9) is the equation does meet OX again, and it is this point that (12) gives us in this case. The equations give us the path not only after projection, but also that path continued backwards, just as if the particle had not started at the point of projection, but had arrived there with the velocity of projection after

travelling from an infinite distance. Sometimes, in solving questions on motion of a particle under gravity, we get negative values of t in our solutions. A negative value of t refers to some point on the parabola produced backwards from the point where the particle was when t was zero.

306. In order to prove general properties of the path of a projectile the simplest method is to assume $v_0 = 0$. For there is a point on the path of every projectile (produced backwards if necessary) at which the vertical velocity is zero. Then the equation connecting x and y becomes

$$y = \frac{g}{2u^2}x^2, \text{ or } \frac{2u^2}{g}y = x^2 \quad . \quad . \quad . \quad . \quad . \quad (13)$$

The origin is now at the vertex. The directrix of this parabola is at a distance $\frac{u^2}{2g}$ above the vertex, and the focus is at the same distance below the vertex.

The kinetic energy of the particle when it is at the vertex is $\frac{1}{2}mu^2$. Also the kinetic energy due to a fall from the directrix to the vertex, a distance $\frac{u^2}{2g}$, is $mg\left(\frac{u^2}{2g}\right) = \frac{1}{2}mu^2$. Thus the kinetic energy at the vertex is that due to a fall from the directrix. If V is the resultant velocity at distance y below the vertex, the energy equation gives

$$\tfrac{1}{2}m\mathrm{V}^2 - \tfrac{1}{2}mu^2 = mgy \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Hence
$$\tfrac{1}{2}m\mathrm{V}^2 = mg\left(y + \frac{u^2}{2g}\right)$$

= work done on the particle in a fall from the directrix . . (15)

If we are given only the equation of the path of a projectile, this last result will enable us to find the velocity at any point of the path.

307. Range on an Inclined Plane.—Let the axis of y be taken vertical and downwards, and the axis of x along the inclined plane. Then all the equations up to (10) in Art. 304 will apply to this case. For, if the vertical acceleration g be resolved along OX and OY, there is no component along OX, because OY is vertical. The equation of the path, referred to these oblique axes, is therefore

$$y = \frac{g}{2u^2}x^2 + \frac{v_0}{u}x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

To find the range on the inclined plane we have only to put $y = 0$, and the value of x which is not zero is the range. Thus the range is

$$x = -\frac{2uv_0}{g} = \frac{2uv'}{g} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

exactly as for the range on a horizontal plane. The difference is, of course, that u and v' are not the same components as when OX is horizontal.

We will put our result in a more convenient form for calculation.

Let the inclined plane make an angle α with the horizontal. Let V_0 be the resultant velocity of projection, and suppose it makes an angle β with the horizontal.

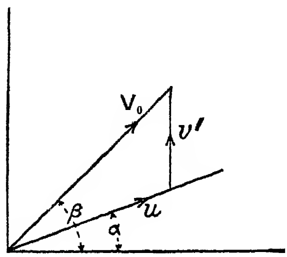


FIG. 156.

From the figure it is clear that

$$u = V_0 \frac{\cos \beta}{\cos \alpha} \quad . \quad . \quad . \quad (3)$$

$$v' = V_0 \frac{\sin (\beta - \alpha)}{\cos \alpha} \quad . \quad (4)$$

Hence the range on the plane

$$= 2V_0^2 \frac{\cos \beta \sin (\beta - \alpha)}{g \cos^2 \alpha} \quad (5)$$

308. From equation (2) or from equation (5) of Art. 307, it is easy to show that there are two different directions of projection with a given velocity V_0 for the same range on the plane, and one of these directions makes the same angle with the vertical as the other makes with the plane.

Putting γ for $(\beta - \alpha)$, the angle which V_0 makes with the plane, the range is

$$2V_0^2 \frac{\cos \beta \sin \gamma}{g \cos^2 \alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Now a body projected with a velocity V_0 at an angle γ with the vertical would have a range

$$2V_0^2 \frac{\cos (90^\circ - \gamma) \sin (90^\circ - \gamma - \alpha)}{g \cos^2 \alpha}$$

$$= 2V_0^2 \frac{\sin \gamma \cos (\gamma + \alpha)}{g \cos^2 \alpha}$$

$$= 2V_0^2 \frac{\sin \gamma \cos \beta}{g \cos^2 \alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

which is the same range as in (5).

It is convenient to remember that the two directions giving the same range make equal angles with the bisector of the angle between the plane and the vertical.

309. Maximum Range on any Plane.—From Fig. 156 it is clear that

$$V_0^2 = u^2 + v'^2 + 2uv' \sin \alpha$$

$$= (u - v')^2 + 2uv'(1 + \sin \alpha) \quad . \quad . \quad . \quad (1)$$

Hence

$$2uv' = \frac{V_0^2 - (u - v')^2}{1 + \sin \alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The range on the inclined plane is thus

$$\frac{V_0^2 - (u - v')^2}{g(1 + \sin \alpha)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If the velocity V_0 is given and the inclination α fixed, this range will be greatest for different directions of projection when $(u - v')^2$ is least; that is, when $u = v'$. This makes the direction of V_0 bisect the angle between the axes. Also the value of the maximum range is

$$\frac{V_0^2}{g(1 + \sin \alpha)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

For a horizontal plane it is only necessary to put $\alpha = 0$.

310. To find the direction of projection of a particle with a given velocity in order that it may pass through a given point.

Let the co-ordinates of the point be (p, q), and suppose α is the inclination of the direction of projection *below* the horizontal. V is the velocity of projection. Then

$$v_0 = V \sin \alpha, \; u = V \cos \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Hence equation (9), Art. 304, becomes, in terms of V and α,

$$y = \frac{g}{2V^2 \cos^2 \alpha} x^2 + x \tan \alpha$$

$$= \frac{g}{2V^2}(1 + \tan^2 \alpha)x^2 + x \tan \alpha \quad . \quad . \quad . \quad . \quad (2)$$

Since the point (p, q) has to lie on this curve, we get

$$q = \frac{g}{2V^2}(1 + \tan^2 \alpha)p^2 + p \tan \alpha \quad . \quad . \quad . \quad . \quad (3)$$

Writing b for $\frac{V^2}{g}$ (which is merely a length), this can be transformed into

$$\tan^2 \alpha + \frac{2b}{p} \tan \alpha + 1 - \frac{2bq}{p^2} = 0 \quad . \quad . \quad . \quad . \quad (4)$$

which is a quadratic equation for $\tan \alpha$. There are two directions of projection as we have already found in Art. 308.

Example.—*A gun can fire a projectile at* 1200 *feet per second. In what direction must it be pointed if the projectile is to hit the top of a building* 100 *feet high at a distance of* 10,000 *feet?*

Here

$$b = \frac{(1200)^2}{32} \text{feet} = 45{,}000 \text{ feet}$$

$$p = 10{,}000, \; q = -100$$

The equation for $\tan \alpha$ is therefore

$$\tan^2 \alpha + 9 \tan \alpha + 1 + 0{\cdot}09 = 0$$

whence $\tan \alpha = -8{\cdot}877$ or $-0{\cdot}123$

L

The angle of elevation is thus

$$-\alpha = 83^\circ\ 34', \text{ or } 7^\circ\ 0', \text{ nearly.}$$

311. Circular Motion with Uniform Speed.—When a body describes a circle of radius r with uniform speed it has an acceleration $\frac{v^2}{r}$ along the radius by Art. 272. If ω is the angular velocity of the radius through the particle, the acceleration can be written $r\omega^2$ since $v = r\omega$.

If the particle makes n revolutions per second, the angular velocity is $2n\pi$ radians per second. Hence the acceleration can also be expressed as $4rn^2\pi^2$. The three expressions for the acceleration are

$$\frac{v^2}{r} = r\omega^2 = 4rn^2\pi^2 \quad \ldots\ldots \quad (1)$$

Suppose a light rod is free to turn about a smooth vertical axis, and suppose it carries a particle of mass m at the end. Let the tension in the rod when it rotates with angular velocity ω radians per sec. be T. Then considering the motion of the particle, we get, by resolving along the inward-drawn radius,

$$T = mr\omega^2 \quad \ldots\ldots \quad (2)$$

In addition to this tension there is, of course, a shearing force and a bending moment in the rod due to the weight of the particle, and these stresses are not affected by the rotation.

312. Conical Pendulum.—A heavy particle is attached by a light string of length l to a fixed point. The particle describes a horizontal circle with the string at an angle α to the vertical. To find the time of a revolution.

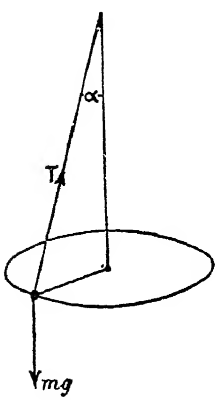


FIG. 157.

The particle describes a circle of radius $l \sin \alpha$. If ω be the angular velocity of the radius through the particle, and T the tension of the string in poundals, we find, by resolving along the radius,

$$ml \sin \alpha \, . \, \omega^2 = T \sin \alpha \quad \ldots \quad (1)$$

and by resolving vertically, since there is no vertical acceleration,

$$mg - T \cos \alpha = 0 \quad \ldots \quad (2)$$

Eliminating T from (1) and (2)

$$mg = ml\omega^2 \cos \alpha \quad \ldots \quad (3)$$

whence

$$\omega^2 = \frac{g}{l \cos \alpha} = \frac{g}{h} \quad \ldots \quad (4)$$

where h is the height of the cone described by the string.

The time for one revolution is

$$\frac{2\pi}{\omega} = 2\pi\sqrt{\frac{h}{g}} \quad \ldots\ldots \quad (5)$$

If α is very small, this is nearly

$$2\pi\sqrt{\frac{l}{g}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and it will be shown later that this is the time for a complete oscillation when the same body swings through a small arc in one plane.

The equations are just the same for a locomotive on a railway curve as for the conical pendulum. But in this case T is the resultant thrust of the rails on the engine. If we write v for the speed of the train instead of $l \sin \alpha . \omega$, equation (1) becomes

$$m\frac{v^2}{r} = \mathrm{T} \sin \alpha$$

where r denotes the radius of the curve. This equation, together with (2), gives

$$\tan \alpha = \frac{v^2}{rg}$$

which gives the angle which the resultant action of the rails on the engine makes with the vertical. Unless the track is banked up at the angle α with the horizontal, the reaction of the engine on the rails will have a component side thrust on the rails.

Suppose the outer rail is raised above the inner rail, the sleepers being inclined at β to the horizontal. Then the lateral thrust of the engine on the rails towards the outside of the curve is equal to the component of $-\mathrm{T}$ up the slope of the sleepers. Let S denote this thrust. Then

$$\begin{aligned} \mathrm{S} &= \mathrm{T} \sin (\alpha - \beta) \\ &= \frac{mg}{\cos \alpha} \sin (\alpha - \beta) \qquad \text{by (2)} \\ &= mg\frac{\sin \alpha \cos \beta - \cos \alpha \sin \beta}{\cos \alpha} \\ &= \mathrm{W}\left(\frac{v^2}{rg} \cos \beta - \sin \beta\right) \end{aligned}$$

where W is written for the weight of the engine.

313. Satellite describing a Circular Orbit about its Primary.—When any body revolves round a much larger body under the gravitational attraction of the larger body, the smaller body is called a satellite to the larger, and this latter is called the primary of the satellite. There are many examples of satellites in the solar system. All the planets are satellites to the sun; the moon is a satellite to the earth. Jupiter has four large satellites and several smaller ones, Saturn has eight, Mars two, Uranus four. Since all these bodies are nearly spherical, they attract, and are attracted, like particles at their centres. In investigating their motions we may, therefore, treat them as particles. We shall assume that the primary remains fixed while the satellite describes a circular orbit. We shall prove later that the larger body does not

remain quite fixed, and it is known from observation that the orbits are nearly circular ellipses.

If m is the mass of the satellite, M that of the primary, r the distance between them, and κ the constant of gravitation, the force acting on the smaller body is $\frac{\kappa \mathrm{M}m}{r^2}$ towards the larger body, that is, along the radius of its circular orbit. The acceleration in the same direction is $r\omega^2$, where ω is the angular velocity. Hence the dynamical equation is

$$mr\omega^2 = \frac{\kappa \mathrm{M}m}{r^2} \quad \dots\dots\dots \quad (1)$$

whence

$$\omega^2 = \frac{\kappa \mathrm{M}}{r^3} \quad \dots\dots\dots \quad (2)$$

If τ is the period of revolution

$$\tau^2 = \left(\frac{2\pi}{\omega}\right)^2 = \frac{4\pi^2}{\kappa \mathrm{M}} \cdot r^3 \quad \dots\dots\dots \quad (3)$$

If there are several satellites to the same primary, M is the same for each satellite. Consequently, for these satellites

$$\tau^2 \propto r^3 \quad \dots\dots\dots \quad (4)$$

This relation between τ and r is a direct consequence of the law of attraction. That is, if the attraction had varied as any other function of r but $\frac{1}{r^2}$, the equation (4) would not have been true. Now it is known from observation of the planets that (4) is true for their motion about their primary the sun. Conversely, it follows from (4) that the force of attraction does vary inversely as the square of the distance, as we will now prove.

314. It is easy to prove from (4) that the force varies inversely as r^2 if we assume the orbits circular. We will give the proof on account of its importance.

$$\tau^2 = \mathrm{C}r^3 \quad \dots\dots\dots \quad (5)$$

Therefore

$$\frac{4\pi^2}{\omega^2} = \mathrm{C}r^3 \quad \dots\dots\dots \quad (6)$$

Hence

$$mr\omega^2 = m\frac{4\pi^2}{\mathrm{C}} \cdot \frac{1}{r^2}$$

$$\propto \frac{1}{r^2} \quad \dots\dots\dots \quad (7)$$

and the left-hand side of (7) is equal to the force on the satellite.

315. Comparison between the Masses of Two Primaries.—Equation (3), Art. 313, combined with observations on the motions of the satellites of different bodies, enables us to find the ratio of the masses of these bodies. From this equation we get

$$\mathrm{M} = \frac{4\pi^2}{\kappa} \cdot \frac{r^3}{\tau^2} \quad \dots\dots\dots \quad (1)$$

If M_1, M_2, are the masses of two bodies, r_1, r_2, the distances of a satellite to each body from its primary, τ_1, τ_2, the periods of these satellites, then

$$M_1 = \frac{4\pi^2}{\kappa} \cdot \frac{r_1^3}{\tau_1^2} \quad \dots \dots \dots \quad (2)$$

$$M_2 = \frac{4\pi^2}{\kappa} \cdot \frac{r_2^3}{\tau_2^2} \quad \dots \dots \dots \quad (3)$$

Hence
$$\frac{M_1}{M_2} = \left(\frac{r_1}{r_2}\right)^3 \left(\frac{\tau_2}{\tau_1}\right)^2 \quad \dots \dots \dots \quad (4)$$

EXAMPLE.—*To compare the masses of the sun and earth, we may take the sun and earth as one system and the earth and moon as the other.*

Then M_1 is the earth's mass, M_2 the sun's mass,

r_1 = moon's distance from earth = 240,000 miles,
r_2 = earth's distance from sun = 92,000,000 miles,
τ_1 = moon's period = $27\frac{1}{3}$ days,
τ_2 = earth's period = $365\frac{1}{4}$ days.

Hence
$$\frac{M_2}{M_1} = \left(\frac{9200}{24}\right)^3 \left(\frac{27\frac{1}{3}}{365\frac{1}{4}}\right)^2$$
$$= 315{,}000 \text{ about} \quad \dots \dots \dots \quad (5)$$

This is not very accurate, because the sun's and moon's distances which we have used contain errors. The following is a more accurate calculation :—

As the result of observations it is found that the earth subtends angles $57' \ 3''$ and $8{\cdot}82''$ at the moon's and sun's mean distances respectively. Then it follows that

$$2r_2 \sin \frac{8{\cdot}82''}{2} = \text{earth's diameter} = 2r_1 \sin \frac{57' \ 3''}{2}$$

Whence
$$\frac{r_2}{r_1} = \frac{\sin \frac{57' \ 3''}{2}}{\sin \frac{8{\cdot}82''}{2}} = \frac{57' \ 3''}{8{\cdot}82''} = \frac{3423}{8{\cdot}82} \text{ nearly} \quad \dots \quad (6)$$

This value of the ratio gives, by (4),

$$\frac{M_2}{M_1} = 327{,}000 \text{ about} \quad \dots \dots \dots \quad (7)$$

316. As one more illustration we will work out the following example :—

At what distance from the earth would a second satellite be if it revolved in exactly one year?

Let r be its distance. Then, using the moon for comparison, equation (4), Art. 313, gives

$$\left(\frac{r}{240{,}000}\right)^3 = \left(\frac{365\frac{1}{4}}{27\frac{1}{3}}\right)^2 = 178{\cdot}6 \quad \dots \dots \quad (1)$$

Therefore
$$r = 1{,}350{,}000 \text{ miles} \quad \dots \dots \dots \quad (2)$$

317. Motion in a Vertical Circle.—A particle P is describing the whole or part of a vertical circle of radius l at the end of a light string or rod attached to the centre. To find the tension in the string and the angular velocity of the particle in any position.

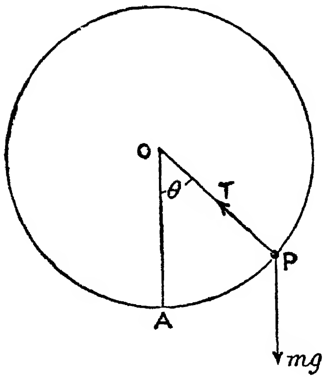


FIG. 158.

Let θ be the angle which the string or rod makes with vertical at any instant. The velocity of the particle is $l\frac{d\theta}{dt}$. Let ω denote the angular velocity of the particle in the lowest position. The work done on the particle from the lowest position up to θ is $-mg(l - l\cos\theta)$. Equating this to the gain in kinetic energy, we get

$$\tfrac{1}{2}ml^2\left\{\left(\frac{d\theta}{dt}\right)^2 - \omega^2\right\} = -mgl(1 - \cos\theta) \quad . \quad . \quad . \quad (1)$$

Hence
$$l^2\left(\frac{d\theta}{dt}\right)^2 = l^2\omega^2 - 2gl(1 - \cos\theta) \quad . \quad . \quad (2)$$

This gives the angular velocity in terms of ω and θ.

Now, the acceleration along PO is $l\left(\frac{d\theta}{dt}\right)^2$. Hence

$$ml\left(\frac{d\theta}{dt}\right)^2 = \mathrm{T} - mg\cos\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Thus
$$\mathrm{T} = mg\cos\theta + ml\left(\frac{d\theta}{dt}\right)^2$$
$$= ml\omega^2 - mg(2 - 3\cos\theta) \text{ by } (2) \quad . \quad . \quad (4)$$

If OP is a rigid rod the particle will describe the complete circle provided $\frac{d\theta}{dt}$ is never zero. Now $\left(\frac{d\theta}{dt}\right)^2$ is least when $\theta = \pi$, and therefore, if it is positive for this value of θ, it will be positive for all values of θ. Hence, the condition that the particle should describe the complete circle is that the right-hand side of (2) should be positive when $\theta = \pi$. This gives

$$ml\omega^2 > 4mg, \text{ i.e. } > \text{four times weight of particle} \quad . \quad . \quad (5)$$

With this condition we find that, in the lowest position, where $\theta = 0$,

$$\mathrm{T} > 4mg + mg, \text{ i.e. } > 5mg \quad . \quad . \quad . \quad . \quad . \quad (6)$$

If, however, OP is a flexible string, the condition that the particle should describe the complete circle is that T should be always positive.

A negative value for T would mean a thrust, and a string cannot supply a thrust. Now T is least at the top, where $\theta = \pi$, and this least value must be positive. That is,

$$ml\omega^2 > 5mg \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

With this condition we find that, at the lowest position,

$$T > 6mg \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

It should be noticed that in all cases the tension in the highest position is less than that in the lowest position by $6mg$. This follows directly from equation (4).

If the angular velocity at the lowest point is too small to satisfy condition (7), then the particle cannot continue on the circle beyond the point at which T becomes zero. But it may happen that $\frac{d\theta}{dt}$ becomes zero before T becomes zero. In the latter case the particle will return along the lower part of the circle, and will, in fact, oscillate about the lowest position. But if T vanishes before the velocity vanishes, then the particle leaves the circle and describes a portion of a parabola until the string becomes tight again.

The critical case in which T and $\frac{d\theta}{dt}$ vanish at the same point is the case when the string comes to rest in the horizontal position. For equation (3) shows that, if T and $\frac{d\theta}{dt}$ are zero at the same time, $\cos\theta$ must be zero, that is, θ must be a right angle. If the string swings above the horizontal position, then, unless the condition (7) is satisfied, the particle will leave the circle.

If a particle slides on the outside of a smooth circular cylinder only an outward thrust is possible. If R is the value of this thrust, we must put $-R$ for T in the preceding equations for motion in a vertical circle. As long as the particle is on the circle R must be positive, and when the equations give a negative value of R, we know that this means that the particle has left the cylinder. Thus, if the particle starts with a negligible velocity in the highest position, equation (2) gives

$$l\omega^2 = 4g \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Hence $$l\left(\frac{d\theta}{dt}\right)^2 = 4g - 2g(1 - \cos\theta) = 2g(1 + \cos\theta) \quad . \quad . \quad (10)$$

and $$-R = mg\cos\theta + 2mg(1 + \cos\theta)$$
$$= mg(2 + 3\cos\theta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

R is zero where $\cos\theta = -\frac{2}{3}$, and below this point it is negative. Consequently the particle will leave the cylinder at this point and begin to describe a parabola in space.

318. Motion on any Smooth Curve under Gravity.—When a particle slides along a smooth curve under the action of any forces whatever, the constraining curve exerts a force perpendicular to the motion

at every instant. Consequently this force does no work. In writing down the energy equation we do not need, therefore, to take any account of the action of the curve.

When the particle is in equilibrium the normal action of the curve is just enough to balance the external forces. The positions of equilibrium are therefore those in which the sum of the components of the external forces along the tangent to the curve is zero. If gravity is the only external force, the equilibrium positions are those points on the curve where the tangents are horizontal.

Suppose the particle is sliding along a smooth curve under the action of gravity only. If the velocity is v at a height y above some fixed horizontal plane, the energy equation is

$$\tfrac{1}{2}mv^2 = -mgy + C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or

$$v^2 + gy = A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Thus the velocity depends only on the height of the particle and the initial velocity, which is involved in A.

Suppose a particle is started from rest at P on the smooth curve in Fig. 159. It will not come to rest again till it reaches Q, the next point

FIG. 159.

at the same height as P. If we measure y upwards from the horizontal plane through P, the energy equation takes the form

$$v^2 + 2gy = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Since y is negative at all points between P and Q, the equation shows that v^2 is positive in this region, and therefore the particle cannot come to rest before reaching Q. After the particle has come to rest at Q, it will fall back till it reaches P again, and then it will begin its career anew. With no air resistance and no friction the particle would go on for ever oscillating between P and Q. Since the speed is the same at the same point of the curve in whichever direction the particle is travelling, the time taken to go from P to Q is the same as that from Q to P. This time is not easy to calculate as a rule.

319. We will give here an expression for the time taken by a particle to slide along a smooth curve in a vertical plane.

Let the equation of the curve referred to rectangular axes OX, OY, (the latter being vertical and positive downwards) be

$$x = F(y) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

makes $f(V) = g$. If u ever reaches V it can certainly never exceed this value. For when $u = V$ the right-hand side of (1) is zero, and consequently u ceases to increase.

If the body starts with a downward velocity greater than V the right-hand side of (1) is negative. Consequently $\frac{du}{dt}$ is negative and u is decreasing, and this will go on until $f(u) = g$, that is, until $u = V$ as before. Thus, whatever be the initial downward velocity the final velocity approaches V, which is found from the equation

$$f(V) = g \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Since $f(u)$ increases with u there can only be one solution of (2). The velocity V is called the *terminal velocity* for the body.

322. There are many examples of bodies having a terminal velocity. One of the most familiar is that of falling snow-flakes. These never reach the ground with a velocity greater than a few feet per second although they may have fallen from a height of several thousand feet. The same thing can be observed in the falling of any light body. If a small feather is dropped from a height of eight feet, it is easy enough to see that it has practically a uniform velocity for the last four or five feet of its fall.

323. It is obvious that the resistance cannot depend on the material of a body. It can only depend on its size and shape as well as its velocity. Hence, if two bodies of the same size and shape but different weights are allowed to fall they will not have the same terminal velocity. If $F(u)$ denotes the whole resistance on each body at velocity u, the resistances per unit mass are

$$\frac{F(u)}{m_1} \quad \text{and} \quad \frac{F(u)}{m_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

respectively, where m_1 and m_2 are the masses of the two bodies. The terminal velocities in the two cases are given by

$$\left.\begin{aligned} F(V_1) &= m_1 g \\ F(V_2) &= m_2 g \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and for the sort of resistance we have been considering this gives the greater terminal velocity to the greater mass. For this reason a lead ball will fall much quicker than a wooden one of the same size, and a solid ball quicker than a hollow one of the same material and size.

324. Influence of Size of Body on Terminal Velocity.—The influence of size on the terminal velocity of a body is very important. Let us consider the motion of a sphere since, on account of symmetry, a sphere may be treated as a particle. It may be assumed that the air resistance is proportional to the area presented to the air, which area is proportional to the square of the radius of the sphere. But the weight of a sphere of given material is proportional to the cube of the radius. Hence the weight and the resistance may be represented by kr^3 and

$r^2F(u)$ respectively, k being a constant for the same material. These two forces are equal when $u = V$, the terminal velocity. Thus

$$F(V) = kr \quad \ldots \quad (1)$$

But since $F(u)$ increases with u, it follows from (1) that the greater r is the greater will be the value of V. Small values of r give small values of V, and that is the reason why small specks of any kind of matter can hang in the air in the form of dust with very little velocity relative to the air. Their terminal velocities are extremely small, and they are acquired in a very short time, so that the slightest air current carries the particles with it.

325. Expression for Resistance with Negative Velocities.—If the resistance per unit mass is $f(u)$ when u is positive, the magnitude of the resistance when u is negative is $f(-u)$. But since reversing the velocity reverses the direction of the resistance the expression for the resistance in the positive direction when u is negative is $-f(-u)$. Thus the equation of motion when the downward velocity u is positive is

$$\frac{du}{dt} = g - f(u) \quad \ldots \quad (1)$$

and when u is negative
$$\frac{du}{dt} = g + f(-u) \quad \ldots \quad (2)$$

These two equations will be the same in particular cases, namely, when

$$f(-u) = -f(u) \quad \ldots \quad (3)$$

in which case $f(u)$ is called an odd function of u. The relation (3) is true for any function containing only odd powers of u; but if there are any even powers in $f(u)$ it will not be true, and in that case a different equation must be used for upward and downward motion.

326. Resistance proportional to Velocity—If the only force besides the resistance is gravity, the equation of motion is

$$\frac{du}{dt} = g - ku \quad \ldots \quad (1)$$

k being a constant, and u the velocity taken positive downwards.

Equation (1) remains unaltered for upward motion provided we regard an upward velocity as negative. When u is negative the force $-ku$ on unit mass is a positive force, that is, a downward force, and that is quite correct because the resistance is contrary to the motion.

From (1) we get

$$\frac{-k}{g-ku}\cdot\frac{du}{dt} = -k \quad \ldots \quad (2)$$

that is,
$$\frac{d}{dt}\cdot\log_e(g-ku) = -k \quad \ldots \quad (3)$$

Hence
$$\log_e(g-ku) = -kt + \log_e C \quad \ldots \quad (4)$$

the constant of integration being written $\log_e C$ for convenience of notation merely.

From (4) we get $\quad g - ku = Ce^{-kt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$

Writing $\frac{dx}{dt}$ for u in this equation and integrating again,

$$kx = gt + \frac{C}{k}e^{-kt} + D \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

If x is the displacement measured from the instant when $t = 0$, we find, on putting x and t zero in (6),

$$D = -\frac{C}{k} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Hence $$kx = gt - \frac{C}{k}(1 - e^{-kt}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Equation (5) shows that $ku = g$ when $t = \infty$. This gives the velocity which we have called the terminal velocity, and it agrees with the result arrived at by general reasoning in Art. 321. Writing V for the terminal velocity $\frac{g}{k}$, equation (8) takes the form

$$\frac{g}{V}x = gt - C\frac{V}{g}\left(1 - e^{-\frac{gt}{V}}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

or $$x = Vt - B\left(1 - e^{-\frac{gt}{V}}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where B is written for the constant $C\frac{V^2}{g^2}$.

If $\frac{dx}{dt} = 0$ when $t = 0$, we find, by differentiating (10) and putting in these values, that $B = \frac{V^2}{g}$. In this case, therefore,

$$x = Vt - \frac{V^2}{g}\left(1 - e^{-\frac{gt}{V}}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Thus suppose the terminal velocity is 8 feet per second, then the displacement from rest is

$$x = 8t - 2(1 - e^{-4t}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

and $$\frac{dx}{dt} = 8 - 8e^{-4t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

At the end of half a second from the start the velocity is 6·92 feet per second, and at the end of two seconds it is 7·997 feet per second. Thus the velocity rapidly approaches its terminal value 8.

If the initial downward velocity was greater than the terminal velocity $\frac{g}{k}$, the constant C which occurs in (5) must be negative. This shows that u is always greater than V, but quickly approaches this value as before.

327. Resistance proportional to the Square of the Velocity.—For downward motion the equation is—

$$\frac{du}{dt} = g - ku^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Writing V for the terminal velocity $\sqrt{\frac{g}{k}}$ (Art. 321), we get, on removing k from (1),

$$\frac{du}{dt} = \frac{g}{V^2}(V^2 - u^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence
$$\frac{1}{V^2 - u^2} \cdot \frac{du}{dt} = \frac{g}{V^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Integrating with respect to t,

$$\frac{1}{2V} \log_e \frac{V + u}{V - u} = \frac{g}{V^2}t + \text{a constant} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Therefore
$$\frac{V + u}{V - u} = Ae^{\frac{2gt}{V}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

from which
$$\frac{u}{V} = \frac{Ae^{\frac{2gt}{V}} - 1}{Ae^{\frac{2gt}{V}} + 1} = \frac{Ae^{\frac{gt}{V}} - e^{-\frac{gt}{V}}}{Ae^{\frac{gt}{V}} + e^{-\frac{gt}{V}}} \quad . \quad . \quad . \quad (6)$$

Writing $\frac{dx}{dt}$ for u and integrating, we find

$$\frac{x}{V} = \frac{V}{g} \log_e \left(Ae^{\frac{gt}{V}} + e^{-\frac{gt}{V}}\right) + \frac{V}{g} \log_e B$$
$$= \frac{V}{g} \log_e B\left(Ae^{\frac{gt}{V}} + e^{-\frac{gt}{V}}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The constant of integration is written $\frac{V}{g}$ $\log_e B$ in (7) for convenience.

When t is very great $e^{-\frac{gt}{V}}$ is very small, and in this case (6) shows that $u = V$ nearly, which verifies that $\sqrt{\frac{g}{k}}$ is the terminal velocity.

If $u = 0$ when $t = 0$, (5) shows that $A = 1$. If also $x = 0$ when $t = 0$, (7) shows that $B = \frac{1}{2}$. Then

$$\frac{u}{V} = \frac{e^{\frac{2gt}{V}} - 1}{e^{\frac{2gt}{V}} + 1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and
$$\frac{x}{V} = \frac{V}{g} \log_e \tfrac{1}{2}\left(e^{\frac{gt}{V}} + e^{-\frac{gt}{V}}\right)$$
$$= \frac{V}{g} \log_e \cosh \frac{gt}{V} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Suppose the terminal velocity is 192 feet per second. Then

$$\frac{u}{V} = \frac{e^{\frac{1}{3}t} - 1}{e^{\frac{1}{3}t} + 1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

When $t = 3$ secs. this gives $\quad u = 0{\cdot}462V = 88{\cdot}7$ feet per sec.,
and when $t = 12$ secs. $\quad u = 0{\cdot}964V = 185$ feet per sec.

When the velocity is negative (*i.e.* upward) the equation of motion is

$$\frac{du}{dt} = g + ku^2 = \frac{g}{V^2}(V^2 + u^2) \quad . \quad . \quad . \quad (11)$$

Hence, on integrating,

$$\frac{1}{V}\tan^{-1}\frac{u}{V} = \frac{g}{V^2}(t - A) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

and therefore

$$\frac{u}{V} = \tan\frac{g}{V}(t - A) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Integrating again to get the downward displacement x,

$$-\frac{x}{V} = \frac{V}{g}\log_e\cos\frac{g}{V}(t - A) + \frac{V}{g}\log_e B$$

$$= \frac{V}{g}\log_e B\cos\frac{g}{V}(t - A) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Since u is negative when $t = 0$, (13) shows that A is positive. When $t = A$, u is zero, and then the body begins to move downward. The equations for downward motion must be used after this.

328. The Path of a Projectile assuming the Resistance to be proportional to the Velocity.—Let v and u be the downward and horizontal components of the velocity at any instant. Since the resistance always acts in the same line as the velocity and is proportional to the velocity, its components parallel to the axes will be proportional to the components of the velocity. That is, the horizontal and vertical resistances are ku and kv respectively.

The equations of motion are therefore, for unit mass,

$$\frac{dv}{dt} = g - kv \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$\frac{du}{dt} = -ku \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Equation (1) is of exactly the same type as (1) in Art. 326. If y is the downward displacement from the instant when t was zero, equation (8) of that article gives

$$y = \frac{g}{k}t - \frac{C}{k^2}(1 - e^{-kt}) \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Again, from (2) $\quad \frac{1}{u}\cdot\frac{du}{dt} = -k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$

Therefore $$\log_e u = -kt + \log_e u_0 \quad . \quad . \quad . \quad (5)$$
where u_0 is the initial horizontal velocity.

Equation (5) gives $$u = \frac{dx}{dt} = u_0 e^{-kt} \quad . \quad . \quad . \quad (6)$$

whence $$x = \frac{u_0}{k}(1 - e^{-kt}) \quad . \quad . \quad . \quad (7)$$

the constant $\frac{u_0}{k}$ being added so as to make $x = 0$ when $t = 0$.

Equation (7) shows that the horizontal displacement can never exceed $\frac{u_0}{k}$. Also from (6) the horizontal velocity rapidly approaches zero unless k is an extremely small quantity.

It should be noticed, also, that the horizontal and the vertical motion are quite independent. The vertical velocity and displacement can be determined completely from the initial vertical velocity without any reference to the horizontal velocity.

Since the vertical velocity quickly approaches the terminal velocity $\frac{g}{k}$, and the horizontal velocity as quickly approaches zero, it follows that after a short time the body is moving nearly vertically with a velocity $\frac{g}{k}$. In fact, the path has a vertical asymptote at a distance $\frac{u_0}{k}$ from the starting-point. We shall denote $\frac{u_0}{k}$ by a.

Eliminating t from (3) and (7), we get the equation to the path—

$$y = \frac{g}{k^2} \log_e \frac{a}{a - x} - \frac{Cx}{ak^2} \quad . \quad . \quad . \quad (8)$$

The constant C depends on the initial vertical velocity; or it can be determined from the direction of projection by equating $\frac{dy}{dx}$ when $x = 0$ to the tangent of the angle of projection below the horizontal.

329. Downward Vertical Motion with Resistance proportional to the n^{th} Power of the Velocity.—We cannot find the displacement in this case, but we can express the time in terms of the velocity by an infinite series which is not very bad to manage.

The equation for unit mass is

$$\frac{du}{dt} = g - ku^n \quad . \quad . \quad . \quad (1)$$

The terminal velocity V is the value of u which makes the resistance equal to g. Thus

$$V^n = \frac{g}{k} \quad . \quad . \quad . \quad (2)$$

CHAPTER XVI

SMALL OSCILLATIONS OF A PARTICLE

330. The Simple Pendulum.—We have already worked out some examples on the oscillation of particles assuming the motion to be rectilinear. We are now going to investigate oscillations allowing for motion in two dimensions.

We shall first work out the motion in a particular but important case, namely, that of a simple pendulum. A simple pendulum is a heavy particle swinging in one plane at the end of a light string or rod. An ideal simple pendulum does not exist, but a ball of lead with a diameter of about half an inch swinging at the end of a light string about a yard long is very nearly an ideal pendulum.

When the string makes an angle θ with the vertical the velocity of the particle is $l\frac{d\theta}{dt}$, l being the length of the string. If m is the mass of the particle, the equation obtained by resolving along the tangent is

$$ml\frac{d^2\theta}{dt^2} = -mg\sin\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now we are dealing with small oscillations, so that we may assume θ to be always small. And for small angles the sine is very nearly equal to the radian measure of the angle. For example, sin 30° is 0·5000 and the radian measure of 30° is 0·524, and 30° is a large angle for pendulum oscillations.

Putting, therefore, θ for $\sin\theta$ in (1), we get

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This type of equation has been solved in Art. 292, and its solution can be written

$$\theta = a\sin\left(\sqrt{\frac{g}{l}}t + \beta\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where a and β are constants of integration.

The time for a complete oscillation is

$$\tau = 2\pi\sqrt{\frac{l}{g}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The constant a is the amplitude of θ. Thus the string swings from an angle a on one side to a on the other side of the vertical.

By observing τ for a simple pendulum and measuring l the equation

(4) will give g. There are more accurate methods of finding g than by the simple pendulum, but a very good value can be obtained by this means. The error need not be more than 0·5 per cent. if the oscillations are small—not more than 10°, say, from the vertical. To get τ the time for fifty or a hundred oscillations should be observed. Then g is calculated from the formula

$$g = \frac{4\pi^2 l}{\tau^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

331. Integration of the Accurate Equation for a Simple Pendulum.—The accurate relation between t and θ at any instant can only be expressed in terms of elliptic integrals. Nevertheless, the period of a complete oscillation can be expressed in an infinite series of powers of $\sin\frac{a}{2}$, which series always converges fairly rapidly.

Multiplying both sides of equation (1) of last article by $\frac{2}{ml}\cdot\frac{d\theta}{dt}$,

$$2\frac{d\theta}{dt}\cdot\frac{d^2\theta}{dt^2} = -2\frac{g}{l}\sin\theta\frac{d\theta}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or

$$\frac{d}{dt}\left\{\left(\frac{d\theta}{dt}\right)^2\right\} = 2\frac{d}{dt}\left(\frac{g}{l}\cos\theta\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Integrating with respect to t,

$$\left(\frac{d\theta}{dt}\right)^2 = 2\frac{g}{l}\cos\theta + C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If a is the amplitude for θ, that is, the value of θ when $\frac{d\theta}{dt} = 0$, we find that $C = -2\frac{g}{l}\cos a$. Whence

$$\left(\frac{d\theta}{dt}\right)^2 = 2\frac{g}{l}\left(\cos\theta - \cos a\right)$$

$$= 2\frac{g}{l}\left(2\sin^2\frac{a}{2} - 2\sin^2\frac{\theta}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Therefore

$$\frac{dt}{d\theta} = \tfrac{1}{2}\sqrt{\frac{l}{g}}\cdot\frac{1}{\sqrt{\left(\sin^2\frac{a}{2} - \sin^2\frac{\theta}{2}\right)}} \quad . \quad . \quad . \quad (5)$$

The time taken by the pendulum from the lowest position to that where $\theta = a$ is clearly a quarter of the complete period. Hence, denoting the complete period by τ, and writing a for $\sin\frac{a}{2}$ for the sake of brevity,

$$\tau = 2\sqrt{\frac{l}{g}}\int_0^a \frac{d\theta}{\sqrt{\left(a^2 - \sin^2\frac{\theta}{2}\right)}} \quad . \quad . \quad . \quad . \quad (6)$$

Now putting $$\sin\frac{\theta}{2} = a\sin\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

we find $$\sqrt{\left(a^2 - \sin^2\frac{\theta}{2}\right)} = a\cos\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and $$\tfrac{1}{2}\cos\frac{\theta}{2}\,d\theta = a\cos\phi d\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Also the limits for ϕ are 0 and $\frac{\pi}{2}$. Hence

$$\sqrt{\frac{g}{l}}\cdot\tau = 2\int_0^{\frac{\pi}{2}} \frac{2d\phi}{\cos\frac{\theta}{2}}$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{2d\phi}{\sqrt{(1 - a^2\sin^2\phi)}}$$

$$= 4\int_0^{\frac{\pi}{2}}\left(1 + \tfrac{1}{2}a^2\sin^2\phi + \frac{1\cdot 3}{2\cdot 4}a^4\sin^4\phi + \ldots\right)d\phi\,. \quad (10)$$

This last step is performed by expanding $(1 - a^2 \sin\phi)^{-\frac{1}{2}}$ by the binomial theorem.

Now it is proved in works on the integral calculus that

$$\int_0^{\frac{\pi}{2}} \sin^{2n}\phi d\phi = \frac{(2n-1)(2n-3)\ldots 3\cdot 1}{2n(2n-2)\ldots 4\cdot 2}\cdot\frac{\pi}{2} \quad . \quad (11)$$

Hence

$$\sqrt{\frac{g}{l}}\cdot\tau = 4\cdot\frac{\pi}{2}\cdot\left(1 + \frac{1}{2}a^2\cdot\frac{1}{2} + \frac{1\cdot 3}{2\cdot 4}a^4\cdot\frac{1\cdot 3}{2\cdot 4} + \ldots\right)$$

$$= 2\pi\left(1 + \frac{1^2}{2^2}a^2 + \frac{1^2\cdot 3^2}{2^2\cdot 4^2}a^4 + \ldots\right) \quad . \quad . \quad . \quad . \quad (12)$$

The value of τ given by the approximate method of the last article is just what we get by omitting all except the first term of the above series.

In order to show the error in the approximate result, we give here the value of τ when $\alpha = 10°$ and when $\alpha = 60°$.

Taking $\alpha = 10°$, then

$$\sin\frac{\alpha}{2} = \cdot 08716$$

and $$\tau = 2\pi\sqrt{\frac{l}{g}}(1\cdot 00191)$$

Taking $\alpha = 60^\circ$,

$$\sin\frac{\alpha}{2} = \tfrac{1}{2}$$

and
$$\tau = 2\pi\sqrt{\frac{l}{g}}(1\cdot073)$$

In the first case the error in taking only the first term is less than 0·2 per cent., and in the second case, for so large an oscillation as 60° from the vertical, the error is only 7 per cent.

332. Small Oscillations of a Particle about a Position of Stable Equilibrium on any Smooth Curve.—The particle P is constrained to move on the smooth curve BAB′. A, a point of minimum height on the curve, is a position of stable equilibrium. If the particle be slightly displaced from A it will fall back to A and then shoot beyond in consequence of the velocity acquired in falling. Then it will fall back again, and thus it will go on oscillating for ever if there is no friction.

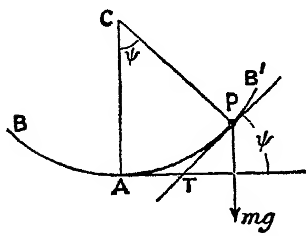


FIG. 160.

AT is the tangent at A and is horizontal, PT is the tangent at P, and PC is the normal at P meeting the normal at A in C. When the angle ψ between the normals at A and P is infinitely small, C becomes the centre of curvature of the curve at A, and AC becomes the radius of curvature.

Denoting the arc AP by s it is clear that, for small values of ψ,

$$s = AC \, . \, \psi \text{ nearly} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If ρ is the radius of curvature at A, there is very little error in putting ρ for AC. Hence

$$s = \rho\psi \text{ nearly} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The dynamical equation for the motion of P along the tangent at P is

$$m\frac{d^2s}{dt^2} = -mg\sin\psi = -mg\psi \text{ nearly}$$

$$= -mg\frac{s}{\rho} \text{ nearly} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

That is,
$$\frac{d^2s}{dt^2} = -\frac{g}{\rho}s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

for small oscillations.

This is the same type of equation as the one for small oscillations of a simple pendulum (Art. 330, equation 2). The solution is

$$s = A\sin\left(\sqrt{\frac{g}{\rho}} \, . \, t + \beta\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and the time of oscillation is

$$\tau = 2\pi\sqrt{\frac{\rho}{g}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Thus the particle oscillates in the same time as a pendulum of length ρ. The simple pendulum is included in the preceding as a particular case.

EXAMPLE 1.—*To find the time of oscillation of a particle about the lowest point of the smooth curve*

$$s = a \sin 4\psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

s being measured from the lowest point.

Here $$\rho = \frac{ds}{d\psi} = 4a \cos 4\psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The value of ρ at the lowest point of the curve, that is, the point where $\psi = 0$, is

$$\rho = 4a$$

Hence $$\tau = 2\pi\sqrt{\frac{4a}{g}} = 4\pi\sqrt{\frac{a}{g}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

EXAMPLE 2.—*To find the time of oscillation about the lowest point of the curve*

$$4ay = x^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

the y-axis being upwards.

The formula for radius of curvature is

$$\rho = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

From (10) $$\frac{dy}{dx} = \frac{x}{2a}, \qquad \frac{d^2y}{dx^2} = \frac{1}{2a} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Putting $x = 0$ in the expression for $\frac{dy}{dx}$ and substituting in (11), we get the value of ρ at the lowest point. Thus

$$\rho = 2a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Hence $$\tau = 2\pi\sqrt{\frac{2a}{g}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

EXAMPLE 3.—*Find the period of oscillation about the minimum point of the curve*

$$6a^2y = 2x^3 - 3x^2a - 36xa^2$$

Here $$6a^2\frac{dy}{dx} = 6x^2 - 6xa - 36a^2$$
$$= 6(x - 3a)(x + 2a)$$

and $$6a^2\frac{d^2y}{dx^2} = 12x - 6a$$

The minimum point is at $x = 3a$, and at this point

$$\frac{d^2y}{dx^2} = \frac{5}{a}$$

and therefore

$$\rho = \frac{a}{5}$$

Consequently the period of oscillation is $2\pi\sqrt{\frac{a}{5g}}$.

333. Isochronous Pendulum.—In the last article the time of a small oscillation of a particle on a smooth curve is calculated. But, just as for the circular pendulum, the time is generally only approximately correct, and for large oscillations it will depend on the amplitude. But there is one type of curve on which the oscillations are absolutely independent of amplitude.

As in the last article, the equation of motion along the tangent is

$$\frac{d^2s}{dt^2} = -g\sin\psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now, if the intrinsic equation of the curve is

$$s = l\sin\psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

then the above equation of motion becomes

$$\frac{d^2s}{dt^2} = -\frac{g}{l}s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and we know that this indicates oscillatory motion with a period

$$\tau = 2\pi\sqrt{\frac{l}{g}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This is exact for small or large oscillations.

Equation (2) is the equation of a cycloid, and this cycloid is the curve described by a point on the circumference of a circle of radius $\frac{1}{4}l$ which rolls on the lower side of a horizontal line. The oscillations on a cycloidal arc are called *isochronous*, because the period is the same for large or small oscillations.

334. The Equation of a Cycloid.—Let PCC′ be a circle of radius a which rolls on the line QC. APM is the curve described by P in this motion.

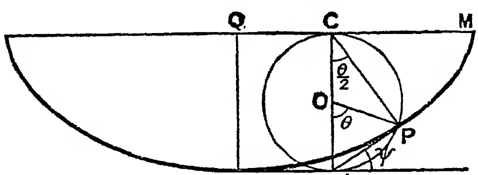


FIG. 161.

Since C is the instantaneous centre of rotation, the direction of motion of P is perpendicular to CP. Hence the tangent to the path of P is PC′, which is perpendicular to CP. It is easy to prove from the figure that $\psi = \frac{\theta}{2}$.

The angular displacement of the circle from the position where OP was vertical is θ. When the circle turns through an additional angle $d\theta$, the displacement of P is CP . $d\theta$, because there is an instantaneous rotation about C. Thus the displacement of P is

$$2a \cos\frac{\theta}{2}\, d\theta = 4a \cos\psi d\left(\frac{\theta}{2}\right)$$

$$= 4a \cos\psi d\psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If ds is the length of the arc described by P in this small motion,

$$ds = 4a \cos\psi d\psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence $$s = 4a \sin\psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

no constant being needed if s is measured from A, where $\psi = 0$.

The student will probably think that CP turns through $d\left(\frac{\theta}{2}\right)$, and therefore that $ds = \text{CP} . d\left(\frac{\theta}{2}\right)$. But CP must be supposed rigidly attached to the circle, and if one line turns through $d\theta$ every other line turns through the same angle. Another way of looking at the question is this: the displacement of P is to the displacement of O as CP is to CO. But the displacement of O is clearly CO . $d\theta$. Hence the displacement of P is CP . $d\theta$.

335. Since perfectly smooth bodies do not exist, it would not be possible to realise the motion we have been investigating by allowing a

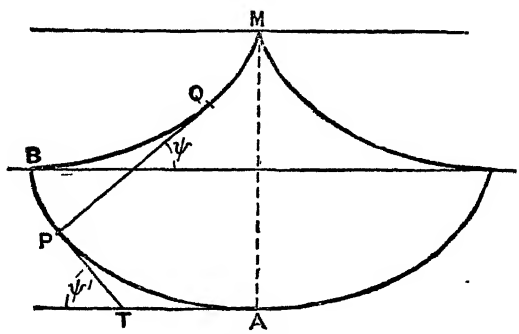


FIG. 162.

particle to slide on any body in the form of a cycloid. But there is a simple mechanical way of removing almost all frictional resistances except that of the air, and this we shall now explain.

Suppose a particle P is carried by a light string of length l, the upper end of which is attached to M, the cusp of two similar cycloids whose equations are $s = l \sin\psi$. The particle oscillates in the plane of the cycloids, and during the motion the string is wrapped on each curve in turn. Under these circumstances P describes a cycloid

exactly equal to each of the guiding curves. Let s', ψ', refer to the curve described by P; s, ψ, to the curve MB, whose equation is $s = l \sin \psi$. At M, where $\psi = \frac{\pi}{2}$, s is equal to l; that is, the arc $BM = l$, the length of the string. Consequently

$$PQ = \text{arc } BQ = l \sin \psi \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

But since the motion of P is perpendicular to QP, we find that

$$\psi' = \frac{\pi}{2} - \psi.$$

Hence

$$PQ = l \cos \psi' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

When ψ' increases by $d\psi'$ the line PQ turns through the same angle $d\psi$ as PT turns through, because these lines are at right angles. If ds' is the arc described by P in this small motion,

$$ds' = PQ \, . \, d\psi' = l \cos \psi' d\psi' \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Hence

$$s' = l \sin \psi' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

is the equation of the curve described by P, provided s' is measured from the lowest point A. And this is the same form of equation as that of each guiding curve.

The pendulum formed in this way would oscillate in the same period for all possible amplitudes.

336. General Method of dealing with Small Oscillations of a Particle about a Position of Stable Equilibrium.—Suppose a particle, acted on by known forces, is constrained to move along a given curve. If there are no frictional resistances the reaction of the constraining curve will be along the normal, and the external forces will generally depend only on the position of the particle. Let s denote the length of the curve measured from some fixed point on it. Assuming that there is no friction, there is a definite tangential force for each value of s. Let therefore $f(s)$ denote the tangential force on unit mass of the particle, the force being taken as positive when it is in the same direction as a positive increment of s and negative in the opposite direction. The dynamical equation for motion along the tangent is thus

$$m\frac{d^2s}{dt^2} = mf(s) \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or

$$\frac{d^2s}{dt^2} = f(s) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Suppose that the point $s = a$ is a position of equilibrium on the curve. Since the tangential force must be zero at a position of equilibrium, it follows that $f(a) = 0$. We shall investigate the motion when the particle is slightly displaced from this equilibrium position.

Write $(a + z)$ for s. Then equation (2) becomes

$$\frac{d^2(a + z)}{dt^2} = f(a + z) = f(a) + zf'(a) + \frac{z^2}{\lfloor 2}f''(a) + \quad . \qquad (3)$$

Since $f(a) = 0$, the largest term on the right-hand side of equation (3) is $zf'(a)$ as long as z is small. Neglecting the higher powers of z, equation (3) becomes

$$\frac{d^2z}{dt^2} = zf'(a) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

for small values of z. If $f'(a)$ is a negative quantity, this equation indicates oscillatory motion with a period

$$\tau = \frac{2\pi}{\sqrt{-f'(a)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Denoting the tangential force on unit mass by T, we have

$$\mathrm{T} = f(s) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Consequently $$f'(s) = \frac{d\mathrm{T}}{ds} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Thus the coefficient of z in (4) is the value of $\frac{d\mathrm{T}}{ds}$ in the equilibrium position.

It should be noticed on what assumptions this solution depends. It has been assumed that the term containing the first power of z is much greater than all the terms containing the higher powers. To ensure that this should always hold we must not only know that z is small initially, but also that the velocity is small. For it is clear that if a particle were passing through an equilibrium position with a large velocity, the displacement z would be large before it came to rest, and our assumption that the term containing z is greater than those containing the higher powers of z would probably be wrong.

Nevertheless, if $f'(a)$ is negative, small oscillations are possible under favourable conditions; and when small oscillations are possible about a position of equilibrium, that position must be one of stable equilibrium. But if $f'(a)$ is positive small oscillations are not possible, because in this case equation (4) has the form

$$\frac{d^2z}{dt^2} = c^2z \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and the solution of this equation is

$$z = \mathrm{A}e^{ct} + \mathrm{B}e^{-ct} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

This does not represent oscillatory motion, and, unless A is zero, which is very improbable in any actual motion, z becomes very large for large values of t. Before this stage is reached the equation (4) has ceased to be approximately correct, for the higher powers of z have become as important as the first power.

337. Since the equation (8) of the last article occurs rather often in dynamics, it is worth while to show that (9) is the solution.

Putting $$z = ue^{ct} \quad . \quad . \quad . \quad . \quad . \quad (1)$$
we find by differentiating twice that
$$\frac{d^2z}{dt^2} = e^{ct}\left(\frac{d^2u}{dt^2} + 2c\frac{du}{dt} + c^2u\right) \quad . \quad . \quad . \quad (2)$$
Hence the equation
$$\frac{d^2z}{dt^2} = c^2z \quad . \quad . \quad . \quad (3)$$
becomes
$$e^{ct}\left(\frac{d^2u}{dt^2} + 2c\frac{du}{dt} + c^2u\right) = c^2ue^{ct}$$
or
$$\frac{d^2u}{dt^2} + 2c\frac{du}{dt} = 0 \quad . \quad . \quad . \quad (4)$$
Integrating at once with respect to t,
$$\frac{du}{dt} + 2cu = \mathrm{D} \quad . \quad . \quad . \quad (5)$$
whence
$$-2c\frac{dt}{du} = \frac{2c}{2cu - \mathrm{D}} \quad . \quad . \quad . \quad (6)$$
Integrating with respect to u,
$$\log_e \mathrm{E} - 2ct = \log_e (2cu - \mathrm{D}) \quad . \quad . \quad . \quad (7)$$
where $\log_e \mathrm{E}$ is written for the constant of integration.

The relation (7) can be transformed into
$$\mathrm{E}e^{-2ct} = 2cu - \mathrm{D} \quad . \quad . \quad . \quad (8)$$
Thus
$$\begin{aligned} z &= ue^{ct} \\ &= \frac{1}{2c}(\mathrm{D}e^{ct} + \mathrm{E}e^{-ct}) \\ &= \mathrm{A}e^{ct} + \mathrm{B}e^{-ct} \quad . \quad . \quad . \quad (9) \end{aligned}$$

338. As an example to illustrate the method of Art. 336 we will take the following problem.

A particle, constrained to move on a fixed smooth curve, is acted on by a force towards a fixed point C *which is not on the curve. Assuming that the magnitude of the force depends only on the position of the particle, to find the time of a small oscillation about a position of stable equilibrium.*

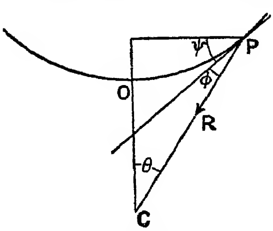


FIG. 163.

If O is a position of stable equilibrium and the force is an attraction towards C, it is clear that CO must be normal to the curve.

Let P be the position of the particle at any instant, and let CP $= r$, CO $= d$, arc OP $= s$. R is the force acting towards C per unit mass.

The equation for motion along the curve is

$$\frac{d^2s}{dt^2} = -\mathrm{R}\cos\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

It may not be easy to express this tangential force $-\mathrm{R}\cos\phi$ in terms of s, but it will be easy enough to find its rate of increase with respect to s in the equilibrium position, and that is all we want. Thus

$$f(s) = -\mathrm{R}\cos\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$f'(s) = +\mathrm{R}\sin\phi\frac{d\phi}{ds} - \frac{d\mathrm{R}}{ds}\cos\phi \quad . \quad . \quad . \quad . \quad (3)$$

Now $\phi = \frac{\pi}{2} - \theta - \psi$. Hence

$$\frac{d\phi}{ds} = -\frac{d\theta}{ds} - \frac{d\psi}{ds} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

At O it is clear that $\frac{d\theta}{ds}$ is $\frac{1}{d}$, and $\frac{d\psi}{ds}$ is $\frac{1}{\rho}$, ρ being the radius of curvature at O. Also $\cos\phi = 0$ at O. Hence

$$f'(0) = -\mathrm{R}\left(\frac{1}{d} + \frac{1}{\rho}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

This is the quantity that corresponds to $f'(a)$ in Art. 336. Hence the period of a small oscillation is

$$\tau = 2\pi\sqrt{\frac{1}{\mathrm{R}\left(\frac{1}{d} + \frac{1}{\rho}\right)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where R and ρ are the force per unit mass and radius of curvature in the equilibrium position.

If the curve is a straight line, then $\rho = \infty$ and $\frac{1}{\rho} = 0$. In this case

$$\tau = 2\pi\sqrt{\frac{d}{\mathrm{R}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

If the force acting on the particle is always in the same direction we need only put $d = \infty$, and then the time becomes

$$\tau = 2\pi\sqrt{\frac{\rho}{\mathrm{R}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

This last result will apply to a particle on a smooth curve under gravity if we put g for R. The expression for τ then agrees with the one found in Art. 332. By means of (6) we can, however, find the time of oscillation of a pendulum, allowing for the fact that gravity does not act in parallel lines, but always towards a fixed point, namely,

the centre of the earth. If a = the earth's radius in feet, and l = length of pendulum also in feet,

$$\tau = 2\pi\sqrt{\frac{1}{g\left(\frac{1}{a}+\frac{1}{l}\right)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

For any ordinary pendulum $\frac{1}{a}$ is insignificant in comparison with $\frac{1}{l}$, so that it is of no use taking account of it.

A body sliding on a perfectly straight line touching the earth's surface may be regarded as a pendulum of infinite length, and its theoretical time of oscillation is

$$2\pi\sqrt{\frac{a}{g}} = 2\pi\sqrt{\frac{4000 \times 1760 \times 3}{32}}\text{secs.} = 85 \text{ mins.} \qquad (10)$$

This is exactly the same as the period of a satellite which would just graze the earth's surface.

339. A horizontal axis, which carries a light rod with a mass attached, is mounted on a vertical axis which rotates with constant angular velocity ω. In consequence of the mechanism and the motion, the mass is free to move along the circumference of a circle which rotates about its vertical diameter. The system constitutes, in fact, a pendulum in a rotating plane. We propose to find the period of oscillation about the position of stable equilibrium relative to the rotating plane.

Let l be the length of the rod, and therefore the radius of the circle described by the particle. When the rod makes an angle θ with the vertical, the distance of the particle from the axis of rotation is $l \sin \theta$. By Art. 273 the acceleration of the particle is the same as if the plane of the circle did not rotate, except that there is an added acceleration $l \sin \theta . \omega^2$ towards the vertical axis and perpendicular to it. The component of this added acceleration along the tangent to the circle is $-l \sin \theta \cos \theta . \omega^2$. Hence, resolving along the tangent to the circle,

$$m\left(\frac{d^2s}{dt^2} - l\sin\theta\cos\theta . \omega^2\right) = -mg\sin\theta \quad . \quad . \quad . \quad (1)$$

or

$$\frac{d^2s}{dt^2} = l\sin\theta\cos\theta . \omega^2 - g\sin\theta \quad . \quad . \quad . \quad . \quad (2)$$

Now suppose the rod is placed at rest relative to the rotating plane at an angle α to the vertical. It will remain in that position if $\frac{d^2s}{dt^2}$ is zero when $\theta = \alpha$, that is, if the right-hand side of (2) is zero. Thus in the positions of relative equilibrium we must have

$$\sin\alpha(l\omega^2\cos\alpha - g) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Whence either

$$\sin\alpha = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

or

$$\cos\alpha = \frac{g}{l\omega^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Equation (4) gives two values of α, namely $\alpha = 0$ and $\alpha = \pi$. The second of these obviously does not correspond to a position of stable equilibrium, for the mass could not be stable at the top of the circle.

We will now examine whether the lowest position is one of stable equilibrium or not.

When θ is small we may write θ for $\sin\theta$ and 1 for $\cos\theta$. Also $s = l\theta$. Hence equation (2) becomes

$$l\frac{d^2\theta}{dt^2} = l\omega^2\theta - g\theta$$
$$= (l\omega^2 - g)\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

If g is greater than $l\omega^2$ the coefficient of θ is negative, and the equation denotes oscillatory motion with a period

$$\tau = 2\pi\sqrt{\frac{l}{g - l\omega^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Thus, when $g > l\omega^2$ the mass is in stable equilibrium in the lowest position; and since $\cos\alpha$ cannot be greater than unity, there is no solution of (5) in this case, and consequently no other position of stable equilibrium.

But when $l\omega^2 > g$, equation (6) does not give oscillatory motion, and now the stable position is given by (5). Suppose now that α is the root of (5). Writing $\alpha + z$ for θ, and $f(\theta)$ for the right-hand side of (2), that equation now becomes

$$l\frac{d^2z}{dt^2} = f(\theta)$$
$$= f(\alpha + z) = f(\alpha) + zf'(\alpha) \quad . \quad . \quad . \quad . \quad (8)$$

for small values of z.

Now,
$$f(\alpha) = l\omega^2\cos\alpha\sin\alpha - g\sin\alpha \quad . \quad . \quad . \quad . \quad (9)$$
$$f'(\alpha) = l\omega^2(2\cos^2\alpha - 1) - g\cos\alpha$$
$$= \frac{g^2}{l\omega^2} - l\omega^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The last line is obtained by substituting for $\cos\alpha$ from (5). The coefficient of z in (8) is therefore

$$-\frac{1}{l\omega^2}(l^2\omega^4 - g^2)$$

which is negative because $l\omega^2 > g$. Also $f(\alpha) = 0$ because α satisfies (3). Consequently equation (8) becomes

$$\frac{d^2z}{dt^2} = -\frac{l^2\omega^4 - g^2}{l^2\omega^2}z \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and this shows that z is a periodic function of t whose period is

$$\tau = \frac{2\pi l\omega}{\sqrt{l^2\omega^4 - g^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Since the mass can oscillate about the position where $\cos\alpha = \frac{g}{l\omega^2}$, it follows that this is the position of stable equilibrium.

If $g = l\omega^2$ the equations (4) and (5) give a common value of α, namely $\alpha = 0$, and this must clearly be the stable position. Moreover, both expressions (7) and (12) agree in giving an infinite time of oscillation; and both are wrong, as common sense tells us, for there must be a finite time of oscillation about a position of stable equilibrium. The results are wrong because they are deduced from an approximate equation in which terms have been neglected which are important in this critical case. When $l\omega^2 = g$ exactly, the first term in the expansion of $f(\alpha + z)$ which does not vanish is the term containing z^3. If this is retained it can be shown that the period of a small oscillation varies inversely as the amplitude β. In fact, the period can be expressed in the form

$$\tau = \frac{\sqrt{2}}{\beta}\cdot 4\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{(1 - \frac{1}{2}\sin^2\phi)}} \qquad (13)$$

The coefficient of $\frac{\sqrt{2}}{\beta}$ in this expression will be found to agree with the accurate formula given in Art. 331 for the time of oscillation of a simple pendulum of length l swinging in a semicircular arc.

Even the above value for τ becomes infinite when we make β infinitely small. To get a reasonable result for τ when $(l\omega^2 - g)$ is very small, but not zero, we ought to take account of the term containing z^3, in which case the time would depend on the amplitude. We are justified in neglecting all higher powers of z than the first in our differential equation only when the terms containing these higher powers are small compared with the term containing z, and this will not generally be true if the coefficient of z is itself small.

340. Forced Oscillations.—A particle which is free to move along a given curve is acted on by a periodic force along the tangent to its path when it is in the neighbourhood of a position of stable equilibrium. To determine the motion when the displacement from the equilibrium position is small.

The problem we are now investigating is one in which, in addition to the periodic force, other forces are brought into play similar to those we have dealt with in small oscillations.

If the periodic force did not act, the equation of motion for small displacements would be of the form

$$\frac{d^2s}{dt^2} = -c^2 s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let $a \sin pt$ be the periodic force on unit mass. Then the equation of motion is

$$\frac{d^2s}{dt^2} = -c^2 s + a\sin pt \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Since $$\frac{d^2}{dt^2}(\sin pt) = -p^2 \sin pt$$

it is obvious that, by a proper choice of H,

$$s = \mathrm{H} \sin pt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

will satisfy (2). Substituting this value of s in (2), we get

$$-p^2\mathrm{H} \sin pt = -c^2\mathrm{H} \sin pt + a \sin pt \quad . \quad . \quad . \quad (4)$$

which is satisfied identically if

$$\mathrm{H} = \frac{a}{c^2 - p^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Hence $$s = \frac{a}{c^2 - p^2} \sin pt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

is a particular solution of (2). But it is not the general solution, because the solution of a differential equation of the second order contains two arbitrary constants, and our present solution does not contain one such constant.

We can get the general solution by putting

$$s = z + \frac{a}{c^2 - p^2} \sin pt \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Then by differentiating

$$\frac{d^2s}{dt^2} = \frac{d^2z}{dt^2} - \frac{p^2a}{c^2 - p^2} \sin pt \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Substituting for s and $\frac{d^2s}{dt^2}$ in (2), we get

$$\frac{d^2z}{dt^2} = -c^2z \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

the terms containing $\sin pt$ having disappeared.

Now, we know the general solution of (9) to be

$$z = \mathrm{A} \sin (ct + \beta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Consequently the general solution of (2) is

$$s = \mathrm{A} \sin (ct + \beta) + \frac{a}{c^2 - p^2} \sin pt \quad . \quad . \quad . \quad . \quad (11)$$

The remarkable feature about this result is that the displacements due to the two forces—the restoring force and the periodic force—are independent of each other. The first term in (11) is exactly the same as if no periodic force acted. The second term is the one due to the periodic force. The amplitude of the second term does, it is true, depend on the period of the natural vibration, because it involves c, but the period is the same as the period of the applied force. The oscillations represented by the first and second terms are called the *free*

oscillations and the *forced oscillations* respectively. In any actual motion the displacements due to the free and forced oscillations are superposed. The character of the motion is best expressed by saying that the particle has free or natural oscillations relative to a framework which moves with the forced oscillations.

Another important point regarding the forced oscillations is that, if $p < c$, the displacement is in the same *phase* as the periodic force; that is, it has the same sign as the force, and reaches its maximum when the force reaches its maximum. But if $p > c$ the displacement is in the opposite phase to the force, for it passes through its minimum as the force passes through its maximum. Now the periods of the free and forced oscillations are $\frac{2\pi}{c}$ and $\frac{2\pi}{p}$ respectively. The preceding statements may therefore be expressed in the following way:—

The forced displacement is in the same phase as the periodic force or in the opposite phase according as the period of the forced oscillations is greater or less than that of the free oscillations.

The periodic force such as we have considered is most easily produced by oscillating the body to which the particle is attached. The next example will illustrate this.

341. *A particle is suspended at the end of an elastic string of negligible mass. The upper end of the string has a simple harmonic vertical oscillation. To find the motion of the particle.*

Let the tension in the string be the product of k and the extension, and let the displacement (downward) of the upper end, O′, of the string be $a \sin pt$. Let OA be the natural length of the string, x the displacement of the particle from the fixed point A.

FIG. 164.

The extension of the string is $(x - a \sin pt)$. Hence the tension is $k(x - a \sin pt)$, and the equation of motion is

$$m\frac{d^2x}{dt^2} = mg - k(x - a \sin pt)$$

$$= -k\left(x - \frac{mg}{k}\right) + ka \sin pt \quad . \quad . \quad . \quad . \quad (1)$$

By putting $y = x - \frac{mg}{k}$ the equation reduces to

$$\frac{d^2y}{dt^2} = -\frac{k}{m}y + \frac{ka}{m} \sin pt \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The solution of this is, by the last article,

$$y = A \sin\left(\sqrt{\frac{k}{m}}t + \beta\right) + \frac{\frac{k}{m}}{\frac{k}{m} - p^2} a \sin pt \quad . \quad . \quad . \quad (3)$$

Now if τ, τ_1, denote the periods of the free and forced oscillations respectively,

$$\tau_1 = \frac{2\pi}{p}, \text{ and } \tau = 2\pi\sqrt{\frac{m}{k}} \quad . \quad . \quad . \quad . \quad (4)$$

Hence the forced oscillations in (3) may be written

$$\frac{\dfrac{1}{\tau^2}}{\dfrac{1}{\tau^2} - \dfrac{1}{\tau_1{}^2}} a \sin pt = \frac{\tau_1{}^2}{\tau_1{}^2 - \tau^2} a \sin pt \quad . \quad . \quad . \quad . \quad (5)$$

Thus, suppose the periods of the free and forced oscillations are 1 second and 2 seconds, then the amplitude of the forced oscillations is $\frac{4}{3}a$. But if the periods are 1 and 1·01, the amplitude is about $50a$, which is very large compared with the amplitude of the motion of O′.

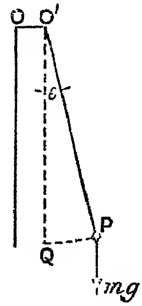


FIG. 165.

342. We will work another example of the same type.

The point of support of a simple pendulum has small horizontal oscillations given by a sin pt. *To determine the motion of the bob.*

Let O be the mean position of the point of support, O′ the position at time t, P the position of the particle. The particle P has the motion of O′ plus the motion relative to O′. Now the acceleration of O′ is

$$\frac{d^2}{dt^2}(a \sin pt) = -ap^2 \sin pt \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and the component of this perpendicular to O′P is

$$-ap^2 \sin pt \cos\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where θ is the angle which O′P makes with the vertical.

The acceleration of P relative to O′ has a component $l\dfrac{d^2\theta}{dt^2}$ perpendicular to O′P. Hence, resolving perpendicular to O′P,

$$m\left(l\frac{d^2\theta}{dt^2} - ap^2 \sin pt \cos\theta\right) = -mg \sin\theta \quad . \quad . \quad . \quad (3)$$

Since θ is supposed to be always small, we may put 1 for $\cos\theta$ and θ for $\sin\theta$. Then equation (3) gives

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta + \frac{ap^2}{l} \sin pt \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The solution of this is

$$\theta = A \sin\left(\sqrt{\frac{g}{l}}\,t + \beta\right) + \frac{a\dfrac{p^2}{l}}{\dfrac{g}{l} - p^2} \sin pt \quad . \quad . \quad . \quad (5)$$

The approximate horizontal displacement of the particle is

$$x = l\theta + a \sin pt = lA \sin\left(\sqrt{\frac{g}{l}}t + \beta\right) + \frac{\frac{g}{l}}{\frac{g}{l} - p^2} a \sin pt \quad . \quad (6)$$

In terms of the periods of the free and forced oscillations τ and τ_1, this can be written

$$x = \mathrm{B} \sin\left(\sqrt{\frac{g}{l}}t + \beta\right) + \frac{\tau_1^2}{\tau_1^2 - \tau^2} a \sin pt \quad . \quad . \quad . \quad (7)$$

We will complete the solution for particular initial conditions. Suppose that the bob starts from rest when $t = 0$. That is, $x = 0$ and $\frac{dx}{dt} = 0$ when $t = 0$. Now,

$$\frac{dx}{dt} = \sqrt{\frac{g}{l}} \mathrm{B} \cos\left(\sqrt{\frac{g}{l}}t + \beta\right) + \frac{\tau_1^2}{\tau_1^2 - \tau^2} ap \cos pt \quad . \quad (8)$$

Putting 0 for x and t in (7),

$$0 = \mathrm{B} \sin \beta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

And putting 0 for $\frac{dx}{dt}$ and t in (8),

$$0 = \sqrt{\frac{g}{l}} \mathrm{B} \cos \beta + \frac{\tau_1^2}{\tau_1^2 - \tau^2} ap \quad . \quad . \quad . \quad . \quad (10)$$

or

$$\mathrm{B} \cos \beta = -ap\sqrt{\frac{l}{g}} \cdot \frac{\tau_1^2}{\tau_1^2 - \tau^2}$$

$$= -a\frac{\tau}{\tau_1} \cdot \frac{\tau_1^2}{\tau_1^2 - \tau^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

From (9) and (11) $\qquad \tan \beta = 0$, *i.e.* $\beta = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$

and

$$\mathrm{B} = -a\frac{\tau}{\tau_1} \cdot \frac{\tau_1^2}{\tau_1^2 - \tau^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

The final value of x is therefore

$$x = \frac{a\tau_1^2}{\tau_1^2 - \tau^2}\left(\sin pt - \frac{\tau}{\tau_1} \sin \sqrt{\frac{g}{l}}t\right) \quad . \quad . \quad . \quad . \quad (14)$$

Suppose, for instance, $\tau = 2$ secs., $\tau_1 = 3$ secs., then

$$\sqrt{\frac{g}{l}} = \frac{2\pi}{\tau} = \pi \quad \text{and} \quad p = \frac{2\pi}{\tau_1} = \tfrac{2}{3}\pi$$

Hence

$$x = \frac{9a}{5}\left(\sin \frac{2\pi t}{3} - \tfrac{2}{3} \sin \pi t\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

But if $\tau = 3$ secs., $\tau_1 = 2$ secs., then

$$x = -\frac{4a}{5}\left(\sin \pi t - \tfrac{3}{2} \sin \frac{2\pi t}{3}\right) \quad . \quad . \quad . \quad . \quad . \quad (16)$$

343. Forced Oscillations when the Period of the Applied Force is the same as that of the Free Oscillations.—The equation is now

$$\frac{d^2s}{dt^2} = -c^2s + a \sin ct \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This cannot be solved by the method of Art. 340, because equation (4) of that article refuses to give a value of H when p and c are equal. We must seek some other kind of solution then.

Without going into the general method of solving such equations as (1) we will show what the solution is in this particular case. The method may seem experimental to the student, but he should not feel dissatisfied at this, because we often have to be content if we can guess the solution of a differential equation. All integration is but judicious guessing based on a knowledge of previous differentiations.

Let us put

$$s = u \cos ct \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where u is a function of t.

Substituting in (1) from (2) we get,

$$\cos ct\frac{d^2u}{dt^2} - 2c \sin ct\frac{du}{dt} - c^2u \cos ct = -c^2u \cos ct + a \sin ct \quad (3)$$

whence . $$\cot ct\frac{d^2u}{dt^2} - 2c\frac{du}{dt} = a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

A particular integral of this is clearly

$$u = -\frac{a}{2c}t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The corresponding value of s is

$$s_1 = -\frac{a}{2c}t \cos ct \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This gives the forced oscillations. We have still to find the general solution giving the free oscillations.

On putting $$s = -\frac{a}{2c}t \cos ct + z \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

n (1) it will be found that the resulting equation for z is

$$\frac{d^2z}{dt^2} = -c^2z \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

the solution of which is

$$z = \text{A} \sin (ct + \beta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Therefore $$s = \text{A} \sin (ct + \beta) - \frac{a}{2c}t \cos ct \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The second term in s represents a displacement with increasing amplitude, and the amplitude is very large when t is large. For small

values of t equation (10) will give approximately correct values of the displacement, but for large values of t they will not even be approximately correct, because, in most cases, the differential equation (1) is an approximate equation obtained by assuming that the terms containing higher powers of s than the first are small compared with the term in s. When this ceases to be true the differential equation, and therefore the integral, are no longer of any use. Thus the equation is only applicable for small values of t. In order to discover what happens when t is large we should have to solve the exact equation of motion, and the form of this will be different for nearly every problem.

344. Small Oscillations of a Particle assuming a Resistance proportional to the Velocity.—If we take the fluid resistance to the motion of a particle oscillating in air or in a liquid to be proportional to the velocity, the equation of motion for small displacements from the position of stable equilibrium is

$$\frac{d^2s}{dt^2} = -c^2s - k\frac{ds}{dt} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or

$$\frac{d^2s}{dt^2} + k\frac{ds}{dt} + c^2s = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This is a linear differential equation the solution of which can be written down immediately by the rules given in works on differential equations. We shall, however, find the solution.

Putting

$$s = e^{-\frac{k}{2}t}z \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and substituting in (2), we get

$$e^{-\frac{k}{2}t}\left\{\frac{d^2z}{dt^2} + \left(c^2 - \frac{k^2}{4}\right)z\right\} = 0 \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Whence

$$\frac{d^2z}{dt^2} + \left(c^2 - \frac{k^2}{4}\right)z = 0 \quad . \quad . \quad . \quad . \quad . \quad (5)$$

If $c^2 > \frac{k^2}{4}$ this is the familiar type of equation solved in Art. 292

Writing b^2 for $c^2 - \frac{k^2}{4}$ for the sake of shortness, the solution of (5) is

$$z = A \sin (bt + \beta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Hence

$$s = Ae^{-\frac{k}{2}t} \sin (bt + \beta) \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

This may be regarded as simple harmonic motion with decreasing amplitude but constant period. The time of a complete oscillation is twice the interval between two successive instants at which the velocity is zero. The velocity is given by

$$\frac{ds}{dt} = Ae^{-\frac{k}{2}t}\left\{-\frac{k}{2} \sin (bt + \beta) + b \cos (bt + \beta)\right\} \quad . \quad . \quad (8)$$

It is easy to see that the interval between successive instants when this is zero is $\frac{\pi}{b}$. Hence the time for a complete oscillation is

$$\frac{2\pi}{b} = \frac{4\pi}{\sqrt{4c^2 - k^2}} \quad \ldots \ldots \ldots \quad (9)$$

The curve showing the relation between s and t is given in Fig. 166. It is clear that the oscillations die out slowly or quickly according as k

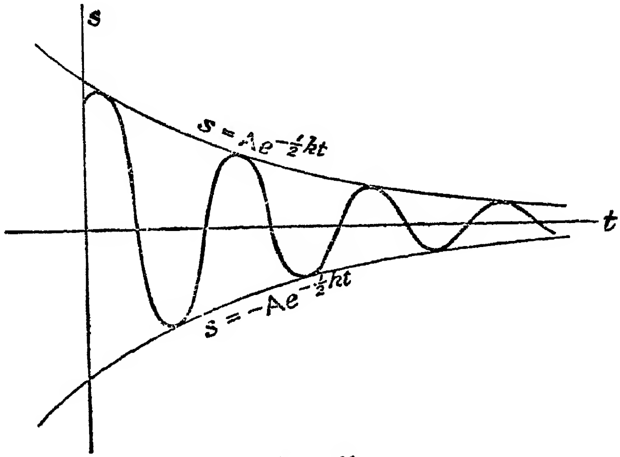


FIG. 166.

is small or large. For if k is small the amplitude $Ae^{-\frac{1}{2}kt}$ diminishes slowly.

If $c = \frac{k}{2}$, the solution of (5) is

$$z = At + B \quad \ldots \ldots \ldots \quad (10)$$

and therefore

$$s = e^{-\frac{k}{2}t}(At + B) \quad \ldots \ldots \ldots \quad (11)$$

Whatever be the initial circumstances of the motion, s ultimately becomes zero, because

$$\lim_{t=\infty} te^{-\frac{k}{2}t} = 0 \quad \ldots \ldots \ldots \quad (12)$$

Moreover, s can only pass once through the value 0, and this will happen if A and B have opposite signs. This cannot be called oscillatory motion; the particle merely falls, in an infinite time, into the equilibrium position, and it may, under favourable circumstances, swing once through the equilibrium position.

We have yet to consider the case where $c^2 - \frac{k^2}{4}$ is negative.

The solution of (5) is now, by Art. 336,

$$z = Ae^{\beta t} + Be^{-\beta t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

where
$$\beta = \sqrt{\frac{k^2}{4} - c^2}$$

Consequently
$$s = (Ae^{\beta t} + Be^{-\beta t})e^{-\frac{k}{2}t}$$
$$= Ae^{\left(\beta - \frac{k}{2}\right)t} + Be^{-\left(\beta + \frac{k}{2}\right)t}$$

or
$$= Ae^{\left(\beta - \frac{k}{2}\right)t}\left(1 + \frac{B}{A}e^{-2\beta t}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Now $\beta - \frac{k}{2}$ is negative, and therefore $e^{\left(\beta - \frac{k}{2}\right)t}$ decreases as t increases, and becomes zero when t is infinite. Hence s approaches zero as t approaches infinity. Also it is seen from (14) that unless $\frac{B}{A}$ is negative and greater than unity, s can never become zero for any finite value of t. Here again we do not get oscillations, but a gradual approach to the equilibrium position with just one possible passage through that position.

345. Forced Oscillations with a Resistance proportional to the Velocity.—If the periodic force per unit mass is $a \sin pt$, the equation of motion is

$$\frac{d^2s}{dt^2} + c^2s + k\frac{ds}{dt} = a \sin pt \quad . \quad . \quad . \quad . \quad (1)$$

Since, on differentiating sines and cosines any number of times we get sines and cosines as a result, we assume, in order to get a particular integral of (1),

$$s = G \sin pt + H \cos pt \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where G and H are constants which have to be determined so that (1) shall be satisfied identically.

Substituting in (1) from (2), we get

$$(-p^2H + pkG + c^2H) \cos pt + (-p^2G - pkH + c^2G) \sin pt$$
$$= a \sin pt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This is satisfied if

$$\left.\begin{aligned}(c^2 - p^2)H + pkG &= 0 \\ (c^2 - p^2)G - pkH &= a\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and

from which
$$\left.\begin{aligned}G &= \frac{(c^2 - p^2)a}{(c^2 - p^2)^2 + p^2k^2} \\ H &= -\frac{pka}{(c^2 - p^2)^2 + p^2k^2}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Thus a particular integral is

$$s_1 = \frac{a}{(c^2 - p^2)^2 + p^2k^2}\{(c^2 - p^2) \sin pt - pk \cos pt\} \quad . \quad (6)$$

If now we put $$s = z + s_1$$
in (1), we get

$$\left(\frac{d^2z}{dt^2} + k\frac{dz}{dt} + c^2z\right) + \left(\frac{d^2s_1}{dt^2} + k\frac{ds_1}{dt} + c^2s_1\right) = a \sin pt \quad . \quad (7)$$

But since s_1 satisfies (1) this equation reduces to

$$\frac{d^2z}{dt^2} + k\frac{dz}{dt} + c^2z = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

which is the equation giving the free oscillations, and it has been solved in the last article.

We may express s_1 in a more convenient form than (6), and one which will exhibit the relation between the forced oscillations given by s_1 and the periodic force.

Let α be an angle determined by

$$\left.\begin{aligned} \cos \alpha &= \frac{c^2 - p^2}{\sqrt{\{(c^2 - p^2)^2 + p^2k^2\}}} \\ \sin \alpha &= \frac{pk}{\sqrt{\{(c^2 - p^2)^2 + p^2k^2\}}} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (9)$$

Then $$s_1 = \frac{a}{\sqrt{\{(c^2 - p^2)^2 + p^2k^2\}}}(\cos \alpha \sin pt - \sin \alpha \cos pt)$$

$$= \frac{a}{\sqrt{\{(c^2 - p^2)^2 + p^2k^2\}}} \sin (pt - \alpha) \; . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The maximum and minimum values of the periodic force occur when pt is an odd multiple of $\frac{\pi}{2}$, whereas the maximum and minimum values of the displacement due to the forced oscillations occur when $(pt - \alpha)$ is an odd multiple of $\frac{\pi}{2}$. Thus the displacement lags behind the force by an interval of time $\frac{\alpha}{p}$.

If $p^2 > c^2$ equations (9) show that α lies between $\frac{\pi}{2}$ and π, and when $p^2 < c^2$ they show that α lies between 0 and $\frac{\pi}{2}$; thus the lag may be anything from zero to $\frac{\pi}{p}$, half the period of the applied force.

The amplitude of the forced oscillations when a resistance acts proportional to the velocity is always finite, and is never even large if a

is not large. This motion is therefore of a distinctly different type from the motion when the resistance is absent, for, in the latter case we found that the amplitude of the forced oscillations became infinite when p became equal to c. When the resistance acts, equations (9) show that if $p = c$, then $\alpha = \frac{\pi}{2}$ and the forced oscillations are represented by

$$s_1 = \frac{a}{pk} \sin\left(pt - \frac{\pi}{2}\right) = -\frac{a}{pk} \cos pt \quad . \quad . \quad . \quad (11)$$

Thus, if the period of the applied force is the same as that of the natural oscillations, the forced oscillations lag exactly a quarter of a period behind the applied force.

The free oscillations are the same as when no periodic force acts, and these have been found in the last article. Whether these take the form given by (7), or by (11), or by (14) of that article, they disappear after a short time and leave only the forced oscillations.

The motion of the pendulum of a clock is an example of forced oscillations. The mechanism is such that the applied force must have the same period as the time of swing of the pendulum. But the force is not, however, accurately represented by a single sine function.

If p is the perpendicular from the origin on the line of motion the moment of momentum is clearly mpv. Thus we have now three expressions for the moment of momentum about O, namely,

$$mpv = m\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) = mr^2\frac{d\theta}{dt} \quad . \quad . \quad . \quad . \quad (3)$$

There is still another convenient way of looking at moment of momentum. When the particle describes a small element ds of the arc of its path, the area of the small triangle with ds as base and the lines joining O to its ends as sides is $\frac{1}{2}pds$. Or again, if $d\theta$ is the increment in θ corresponding to ds, the area of the triangle is $\frac{1}{2}r^2d\theta$. Denoting this area by dA, we have

$$dA = \frac{1}{2}pds = \frac{1}{2}r^2d\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Therefore, on dividing by dt

$$\frac{dA}{dt} = \frac{1}{2}p\frac{ds}{dt}\left(= \frac{1}{2}pv\right) = \frac{1}{2}r^2\frac{d\theta}{dt} \quad . \quad . \quad . \quad (5)$$

Thus the moment of momentum can also be written

$$2m\frac{dA}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where A denotes the area swept out by the radius vector from any given instant.

348. Equations of Motion in Polar Co-ordinates.—Let R and T be the radial (outward) and transverse forces acting on the particle of mass m. Using the expressions for acceleration in Art. 271, the equations of motion are

$$m\left\{\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right\} = R \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$m\frac{1}{r}\cdot\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = T \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Multiplying both sides of (2) by r, we may write

$$\frac{d}{dt}\left(mr^2\frac{d\theta}{dt}\right) = rT \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The quantity in the bracket in (3) is the moment of momentum of the particle about the pole, and rT is the moment of the resultant force about the same point, since the moment of R about O is zero. Now since any point may be taken as pole, equation (3) shows that

$$\left.\begin{array}{r}\text{rate of increase of moment of}\\ \text{momentum about any point}\end{array}\right\} = \begin{array}{l}\text{moment of forces about}\\ \quad\text{the same point}\end{array} \quad . \quad . \quad . \quad (4)$$

This is a very important and useful equation, and particularly in dealing with central forces. It is worth while to give another proof.

The equations of motion in cartesian co-ordinates are

$$m\frac{d^2x}{dt^2} = X \quad \ldots \quad (5)$$

$$m\frac{d^2y}{dt^2} = Y \quad \ldots \quad (6)$$

Multiplying these by $-y$ and x respectively and adding, we get

$$m\left(x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2}\right) = xY - yX \quad \ldots \quad (7)$$

Now the right-hand side of (7) is the moment, about the origin, of the force whose components are X and Y. It will be found on differentiating that

$$\frac{d}{dt}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) = x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} \quad \ldots \quad (8)$$

Hence the left-hand side of (7) is the rate of increase of

$$m\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)$$

which is the moment of the momentum of the particle about the origin. Thus the quantities in equation (7) are exactly the same as those in the corresponding sides of equation (3).

349. Particular Case of Central Forces.—If the force acting on the particle always passes through the pole, then $T = 0$, and equation (3) of the last article becomes

$$\frac{d}{dt}\left(mr^2\frac{d\theta}{dt}\right) = 0 \quad \ldots \quad (1)$$

Whence $$r^2\frac{d\theta}{dt} = h, \text{ a constant} \quad \ldots \quad (2)$$

Thus for a central force through the origin, the moment of the velocity (which is the moment of momentum divided by m) is constant. That is,

$$r^2\frac{d\theta}{dt} = pv = x\frac{dy}{dt} - y\frac{dx}{dt} = 2\frac{dA}{dt} = h \quad \ldots \quad (3)$$

This tells us that the angular velocity varies inversely as the square of the radius vector r, and that the area swept out by the radius vector per second is constant and equal to $\frac{1}{2}h$.

350. To find the path described by a particle under a central force which is a function of r only.

Let the centre be taken as pole, and let the attraction per unit mass be P. Let ϕ be the angle between the tangent and the radius vector. To be quite definite,

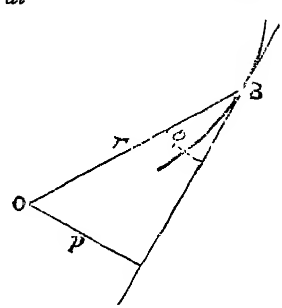


FIG. 167.

ϕ is the angle between the outward-drawn radius vector and the direction of motion.

Resolving along the tangent,

$$\frac{dv}{dt} = -\mathrm{P}\cos\phi = -\mathrm{P}\frac{dr}{ds} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

But
$$\frac{dv}{dt} = \frac{dv}{ds}\cdot\frac{ds}{dt} = v\frac{dv}{ds} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence
$$v\frac{dv}{ds} = -\mathrm{P}\frac{dr}{ds} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Integrating with respect to s,

$$\tfrac{1}{2}v^2 = -\int \mathrm{P}\frac{dr}{ds}ds + \mathrm{C} = -\int \mathrm{P}dr + \mathrm{C} \quad . \quad . \quad . \quad (4)$$

Also, because the origin is at the centre of force,

$$pv = h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Substituting in (4) the value of v from (5), we get

$$\tfrac{1}{2}\frac{h^2}{p^2} = -\int \mathrm{P}dr + \mathrm{C} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This equation gives a relation between p and r for the curve described when P is given. Such a relation is an equation to the curve; it is called the tangential-polar equation to the curve. It is not a very convenient form of equation because it would be rather laborious drawing curves from tangential polar equations even if it could be done at all But we can easily deduce the ordinary polar equation from the tangential-polar equation. We shall show how this is done when we come to find the polar equation by using the polar equations of motion. For the present we give the tangential-polar equations of several well-known curves which are important in this subject.

351. Tangential-polar Equations.

Ellipse with pole at centre, axes $2a$, $2b$,

$$\frac{a^2b^2}{p^2} = a^2 + b^2 - r^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (\alpha)$$

Ellipse with pole at one focus, axes $2a$, $2b$,

$$\frac{b^2}{ap^2} = \frac{2}{r} - \frac{1}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\beta)$$

Hyperbola with pole at one focus,

$$\frac{b^2}{ap^2} = \pm\frac{2}{r} + \frac{1}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\gamma)$$

The upper sign gives the branch of the curve nearer the pole, and the lower sign gives the other branch.

A parabola may be regarded as an ellipse or hyperbola with infinite axes but finite latus rectum. The latus rectum is $2\frac{b^2}{a}$. We can there-

fore get the equation of a parabola by writing l for the semi latus rectum and putting ∞ for a in either (β) or (γ) with the upper sign.

Thus the equation of a parabola with pole at focus is

$$\frac{l}{p^2} = r \quad \ldots\ldots\ldots\ldots \quad (\delta)$$

Circle with pole at centre,

$$p = r \quad \ldots\ldots\ldots\ldots \quad (\epsilon)$$

Circle of radius a with pole on circumference,

$$p = \frac{r^2}{2a} \quad \ldots\ldots\ldots\ldots \quad (\zeta)$$

Equiangular spiral with pole at centre, the angle between the radius vector and the tangent being α,

$$p = r \sin \alpha \quad \ldots\ldots\ldots\ldots \quad (\eta)$$

352. Orbits for Particular Laws of Force.—Suppose the attraction varies inversely as the cube of the distance. Then

$$\mathrm{P} = \frac{k}{r^3} \quad \ldots\ldots\ldots\ldots \quad (1)$$

Equation (6) of Art. 350 gives

$$\frac{1}{2}\cdot\frac{h^2}{p^2} = -\int \mathrm{P}dr + \mathrm{C}$$

$$= +\frac{1}{2}\cdot\frac{k}{r^2} + \mathrm{C} \quad \ldots\ldots\ldots\ldots \quad (2)$$

If C happens to be zero, this takes the form

$$p = \frac{h}{\sqrt{k}} r \quad \ldots\ldots\ldots\ldots \quad (3)$$

which is the equation of an equiangular spiral. By comparing with (η), Art 351, we see that

$$\sin \alpha = \frac{h}{\sqrt{k}} \quad \ldots\ldots\ldots\ldots \quad (4)$$

If $\sin \alpha = 1$, that is, if $k = h^2$, the spiral becomes a circle.

The spiral is not the general orbit for this law of force. It is only in the particular case when $\mathrm{C} = 0$ that we get the spiral orbit. The character of the orbit in the general case will be more easily understood in polar co-ordinates, which we shall come to shortly.

353. Attraction proportional to Distance from Centre.

Here

$$\mathrm{P} = kr \quad \ldots\ldots\ldots\ldots \quad (1)$$

$$\frac{1}{2}\cdot\frac{h^2}{p^2} = -\tfrac{1}{2}kr^2 + \mathrm{C} \quad \ldots\ldots\ldots\ldots \quad (2)$$

Dividing by $\frac{1}{2}k$, we get

$$\frac{h^2}{k}\cdot\frac{1}{p^2} = \frac{2\mathrm{C}}{k} - r^2 \quad \ldots\ldots\ldots\ldots \quad (3)$$

This is of the same form as equation (α), Art. 351. It is therefore the equation of an ellipse with pole at the centre. The semi-axes are given by

$$a^2b^2 = \frac{h^2}{k} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$a^2 + b^2 = \frac{2C}{k} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

which are obtained by a comparison with (α), Art. 351.

From equations (4) and (5) we get

$$a^2 = \frac{1}{k}\{C + \sqrt{(C^2 - kh^2)}\} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$b^2 = \frac{1}{k}\{C - \sqrt{(C^2 - kh^2)}\} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

It might appear from these equations that it would be possible for a^2 and b^2 to be imaginary. For if C^2 is not greater than kh^2 the expressions for a^2 and b^2 will be imaginary, and this would indicate that the orbit is some other curve than the ellipse with pole at the centre. But C^2 must of necessity be greater than kh^2 for any real orbit. For, from (3),

$$2C = \frac{h^2}{p^2} + kr^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Now p must be less than or equal to r, and therefore

$$2C \geqq \frac{h^2}{r^2} + kr^2$$

or

$$2C \geqq \left(\frac{h}{r} - \sqrt{k}r\right)^2 + 2h\sqrt{k} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

It is obvious from (9) that C cannot be less than $h\sqrt{k}$, and therefore the values of a^2 and b^2 are real.

Since the area described by the radius vector per second is $\frac{1}{2}h$, and the whole area of the ellipse is πab, the time taken to describe the whole ellipse is

$$\tau = \frac{\pi ab}{\frac{1}{2}h} = \frac{2\pi ab}{\sqrt{k}\,.\,ab} = \frac{2\pi}{\sqrt{k}} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The motion just investigated is called *elliptic harmonic motion*. It is, in fact, the resultant of two simple harmonic motions of equal period in intersecting lines, the mean positions for the two motions being the point of intersection. Also there is a difference of a quarter of a period between their phases, so that one displacement is a maximum when the other is zero.

354. The Same Problem in Cartesian Co-ordinates.—The components of the attraction kr parallel to OX and OY, whether these

axes are perpendicular or not, are $-kx$ and $-ky$. Hence the equations of motion are

$$\frac{d^2x}{dt^2} = -kx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{d^2y}{dt^2} = -ky \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The solutions of these equations can be written

$$x = \mathrm{A} \cos \sqrt{k}t + \mathrm{B} \sin \sqrt{k}t \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$y = \mathrm{G} \cos \sqrt{k}t + \mathrm{D} \sin \sqrt{k}t \quad . \quad . \quad . \quad . \quad . \quad (4)$$

After eliminating t we get

$$(\mathrm{D}x - \mathrm{B}y)^2 + (\mathrm{G}x - \mathrm{A}y)^2 = (\mathrm{AD} - \mathrm{BG})^2 \quad . \quad . \quad (5)$$

which is plainly the equation to an ellipse since it has the form for a central conic, and the terms containing x and y have no real factors, and this latter is the condition for an ellipse.

Since equations (1) and (2) are true for all axes through the centre of force, we will choose the axes in a convenient position.

Let C be the position of the particle at any instant, and let OC be taken as axis of x. Let OC $= a$. Let the axis of y be parallel to the tangent at C, and let the velocity at C be v. With these conditions we get, measuring t from the instant when the particle was at C,

$$\left.\begin{aligned} \frac{dx}{dt} &= 0 \qquad & x &= a \\ \frac{dy}{dt} &= v \qquad & y &= 0 \end{aligned}\right\} \text{when } t = 0 \quad . \quad . \quad . \quad . \quad (6)$$

Introducing these conditions in (3) and (4), we find

$$\left.\begin{aligned} \mathrm{B} &= 0 \qquad & \mathrm{A} &= a \\ \mathrm{D} &= \frac{v}{\sqrt{k}} \qquad & \mathrm{G} &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Hence

$$\left.\begin{aligned} x &= a \cos \sqrt{k}t \\ y &= \frac{v}{\sqrt{k}} \sin \sqrt{k}t \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (8)$$

Thus

$$\frac{x^2}{a^2} + \frac{ky^2}{v^2} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

This is the equation of an ellipse referred to conjugate diameters as axes, and the length of the semi-diameters are a and $\frac{v}{\sqrt{k}}$. If we write b for the semi-diameter conjugate to a, then

$$v = \sqrt{k}b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

This shows that the velocity at any point is proportional to the semi-diameter parallel to the velocity. This semi-diameter therefore represents the velocity on some scale. Consequently the actual orbit of the particle is the hodograph of its motion.

355. Examples of Elliptic-Harmonic Motion.

EXAMPLE 1.—*An ordinary pendulum, whose motion is not confined to one vertical plane, has approximately simple harmonic motion provided the string always makes small angles with the vertical.*

Let T be the tension in the string, θ its inclination to the vertical at any instant, l its length; and let r be the distance of the bob from the vertical through the point of suspension, z its distance above the lowest point.

Now, $$z = l(1 - \cos\theta) = \frac{l\theta^2}{2} \text{ nearly} \quad . \quad . \quad . \quad . \quad (1)$$

because $$\cos\theta = 1 - \frac{\theta^2}{\lfloor 2} + \frac{\theta^4}{\lfloor 4} + \text{etc.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Since θ is a small angle, we shall neglect all powers of θ beyond the first. With this approximation z is always taken as zero, and then the equation for vertical motion gives

$$\mathrm{T}\cos\theta - mg = m\frac{d^2z}{dt^2} = 0 \text{ nearly} \quad . \quad . \quad . \quad . \quad (3)$$

Hence, again putting 1 for $\cos\theta$,

$$\mathrm{T} = mg \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Now the horizontal force on the bob acts towards the vertical through the point of suspension, and its value is

$$\mathrm{T}\sin\theta = \mathrm{T}\frac{r}{l} = mg\frac{r}{l} \text{ nearly} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Thus the horizontal force is proportional to the horizontal displacement from the equilibrium position. Consequently the horizontal motion is elliptic-harmonic. That is, the plan of the path of the particle is very nearly an ellipse. The time taken by the bob to describe the complete ellipse is

$$\tau = 2\pi\sqrt{\frac{l}{g}} \text{ nearly} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

just as for a simple pendulum of length l.

The conical pendulum (see Art. 312) is included in the above. In this case the ellipse is a circle, and θ, and therefore z, are constant, so that (3) is true exactly.

EXAMPLE 2.—*The ends of an elastic string are attached to two points in the same vertical line, and a particle is attached to the mid-point of the string.*

If l is the length of the string, and T the tension in the vertical position, the force acting on the particle when it is displaced a distance r from the vertical is $2\mathrm{T}\frac{r}{l}$. Thus the motion is elliptic-harmonic, and if m is the mass of the particle, the period is

$$\tau = 2\pi\sqrt{\frac{ml}{2\mathrm{T}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

which is, of course, the same as when the particle oscillates in one vertical plane.

EXAMPLE 3.—*A heavy particle is attached to the free end of a light rod with a uniform circular section, and the other end of the rod is fixed vertically.*

Neglecting the small amount of shear produced by the weight of the particle and of the rod, we find that, when the free end of the rod is displaced from the vertical, there is a constant shearing force all along the rod. If y denotes the displacement at distance x from the fixed end,

$$EI\frac{d^3y}{dx^3} = -F, \text{ a constant} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Integrating this with the conditions that the bending moment is zero where $x = l$ (the length of the rod), and $\frac{dy}{dx}$ and y are each zero where $x = 0$, we get

$$EIy = F(\tfrac{1}{2}lx^2 - \tfrac{1}{6}x^3) \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Putting r for the value of y at the end where $x = l$,

$$EIr = \tfrac{1}{6}l^3F \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

whence

$$F = \frac{6EI}{l^3}r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

This shearing force is the force exerted by the particle on the rod, pulling it away from the vertical. The reaction to this force is a force of equal magnitude exerted by the rod on the particle, pulling the particle *towards* the vertical. Since the force on the particle is proportional to the displacement r, the motion is elliptic-harmonic, and its period is

$$2\pi\sqrt{\frac{ml^3}{6EI}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

since the force on unit mass is $\frac{6EI}{ml^3}r$.

356. The Inverse Square Law.—This is the law of gravitational attraction. In this case

$$P = \frac{k}{r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and the energy equation, with $\frac{h}{p}$ for v, gives

$$\frac{1}{2}\cdot\frac{h^2}{p^2} = -\int P dr + C$$

$$= \frac{k}{r} + C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where C is a positive or negative constant.

Equation (2) can be written

$$\frac{h^2}{k}\cdot\frac{1}{p^2} = \frac{2}{r} + \frac{2C}{k} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If C is positive, this is the equation of a branch of an hyperbola referred to the nearer focus as pole.

On comparing with (γ), Art. 351, we find

$$\frac{b^2}{a}=\frac{h^2}{k} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$\frac{1}{a}=\frac{2C}{k}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The constants h and C can be found from the state of motion at any instant, and then a and b can be determined from (4) and (5).

If, however, C is zero, the path is a parabola, for the equation is the same as (δ), Art. 351. By a comparison of (δ) with (3) when C = 0, we get

$$l=\frac{h^2}{k} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

If we are given the position and the direction of motion at any

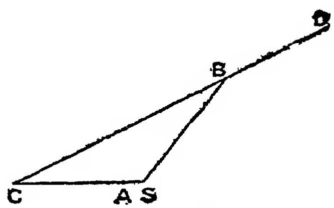


Fig. 168.

instant of a body describing a parabolic path about a focus, it is very easy to construct the parabola.

Let S be the focus, B the position of the particle, CBD the line of motion. Let C be a point such that SC = SB. Then SC is the axis of the parabola, and the vertex A is found by making SA equal to $\frac{1}{4}l$, that is, $\frac{1}{2}\cdot\frac{h^2}{k}$. That SC is the axis follows from the fact that the tangent to a parabola makes equal angles with the axis and the focal distance.

Putting pv for h again in the equation to the parabola, we get

$$\tfrac{1}{2}v^2=\frac{k}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

In a circular orbit of radius r we can easily show by the method of Art. 313 that

$$v^2=\frac{k}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Thus for any given distance r from the attracting centre there is a definite parabolic velocity as well as a definite circular velocity, and the parabolic velocity is $\sqrt{2}$ times the circular velocity at the

same distance. A body which has the parabolic velocity at any point will describe a parabola whatever be its direction of motion. A similar statement is obviously not true for the circular velocity. In order that a body may describe a circle, it must have the circular velocity given by (8), and must be moving at right angles to the radius vector.

A body describing a parabola goes to infinity, and when $r = \infty$ equation (7) shows that $v = 0$. That is, a body describing a parabola has just enough kinetic energy to carry it to infinity and none to spare.

357. Inverse Square Law continued.—If C is negative in (3) of the last article the equation is that of an ellipse. Put C′ for −C, so that C′ is a positive quantity. Then a comparison with (β), Art. 351, gives

$$\frac{b^2}{a} = \frac{h^2}{k}; \quad \frac{1}{a} = \frac{2C'}{k} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

whence

$$\frac{b^2}{a^2} = \frac{h^2}{k} \times \frac{2C'}{k} = \frac{2C'h^2}{k^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Equations (1) determine a and b when h and C′ are known, but the ellipse will not be real unless $a > b$, that is, unless $e^2 > 0$. In order to show that a real ellipse is certainly described if C′ is positive we need to show that e is real.

Now

$$e^2 = \frac{a^2 - b^2}{a^2} = 1 - \frac{2C'h^2}{k^2}$$

$$= 1 - \frac{h^2}{k^2}\left(2\frac{k}{r} - \frac{h^2}{p^2}\right)$$

$$= \left(1 - \frac{h^2}{kp}\right)^2 + 2\frac{h^2}{k}\left(\frac{1}{p} - \frac{1}{r}\right), \quad . \quad . \quad . \quad . \quad (3)$$

which is certainly positive since $p < r$. Thus (3) shows that e^2 is positive and (2) shows that $(1 - e^2)$ is positive, and consequently an ellipse is not only a possible orbit but is the *only* possible orbit.

We may express e^2 in another, and perhaps more interesting, form in terms of the kinetic energy and the potential. Thus let

$$K = \tfrac{1}{2}v^2 = \tfrac{1}{2}\frac{h^2}{p^2}; \quad V = \frac{k}{r}.$$

Then

$$h^2 = 2p^2K \text{ and } k = rV.$$

So, since

$$C' = V \quad K,$$

we get

$$1 - e^2 = \frac{b^2}{a^2} = \frac{4p^2K}{r^2V^2}(V - K) = \frac{p^2}{r^2V^2}\{V^2 - (V - 2K)^2\}.$$

Therefore

$$e^2 = 1 - \frac{p^2}{r^2} + \frac{p^2}{r^2}\left(1 - \frac{2K}{V}\right)^2$$

$$= \cos^2\phi + \left(1 - \frac{2K}{V}\right)^2 \sin^2\phi,$$

where ϕ denotes the angle between the velocity v and the radius vector r, that is, the angle shown in Fig. 167. Thus e is always positive and reduces to zero only when $\phi = 90^\circ$ and $2K = V$. These two conditions mean that the particle is travelling perpendicular to the radius vector and has just the correct velocity for circular motion at that distance.

Since $\frac{1}{2}h$ is the area described per second by the radius vector from the attracting centre to the body in any orbit whatever, it follows that the time occupied by the body in travelling over any portion of its orbit is equal to the area of the sector bounded by that portion of the orbit and the radii vectores through the ends of it divided by $\frac{1}{2}h$. In short, if A is the area swept out by the radius vector in any time t, then

$$t = \frac{A}{\frac{1}{2}h} = \frac{2A}{h} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If τ is the time taken by a body to describe a complete ellipse, the equation (3) gives, since the whole area $= \pi ab$,

$$\tau = \frac{2\pi ab}{h}$$

$$= \frac{2\pi}{\sqrt{k}} a^{\frac{3}{2}} \text{ by (1)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If several bodies describe elliptic orbits about the same attracting mass (assumed to be fixed), then k is the same for each body because it depends only on the attracting mass. Hence for each body

$$\frac{\tau^2}{a^3} = \frac{4\pi^2}{k} = \text{a constant} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Since the time taken to describe an ellipse depends only on the major axis, it will be unaffected by making the minor axis as small as we please. Let, then, the minor axis become infinitely small. In this case the ellipse is infinitely thin with its foci at the ends. Thus a body falling in the centre of force along a straight line may be considered to be describing an infinitely thin ellipse. Hence the time taken to fall from rest at a distance d into the centre of force is half the periodic time in an ellipse of major axis d, namely,

$$\frac{\pi\left(\frac{d}{2}\right)^{\frac{3}{2}}}{\sqrt{k}} = \frac{\pi d^{\frac{3}{2}}}{2\sqrt{2k}}$$

which agrees with the result in Art. 298.

358. Inverse Square Law continued.—The energy equation for the inverse square law is

$$\tfrac{1}{2}v^2 = \frac{k}{r} + C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and it has been shown that the orbit is an ellipse, a parabola, or an hyperbola, according as C is negative, zero, or positive.

Now $\frac{k}{r}$ is the work done by the attracting force in bringing unit mass from infinity to a distance r from the centre of attraction. The negative of this is the potential energy of unit mass at r due to the attraction. This makes the potential energy zero at infinity, and negative at all finite distances from the centre. If the force of attraction is the gravitational attraction of a mass situated at the centre, then $-\frac{k}{r}$ is the quantity which we called the potential of the attracting mass, except that we changed its sign and introduced a constant factor for convenience. It follows that C is the total energy of unit mass of the attracted body, that is, the sum of its kinetic and potential energies, assuming that the potential energy is zero at infinity. The form of the orbit therefore depends on whether the total energy—which remains constant for any given body—is negative, zero, or positive. If the total energy is zero or positive, the body goes to infinity along a parabola or an hyperbola. But if the total energy is negative, the body cannot get to infinity, the place where the potential energy is zero, for this would require the kinetic energy to be negative, which is impossible.

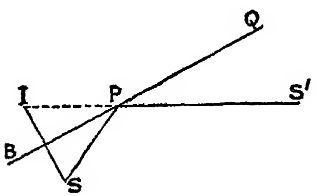


FIG. 169.

359. Construction of an Elliptic Orbit with given Conditions.—Suppose a body is projected along the line PQ, in either direction, with a velocity V. Let S be the attracting centre, and let SP, the initial distance of the attracted body, be denoted by d. The condition for an ellipse is that $\frac{1}{2}V^2$ should be less than $\frac{k}{d}$. It simplifies the work considerably if we suppose $\frac{1}{2}V^2$ expressed in terms of $\frac{k}{d}$. Let, therefore, $\frac{1}{2}V^2 = n\frac{k}{d}$.

The equation which gives an ellipse is

$$\tfrac{1}{2}v^2 = \frac{k}{r} - \frac{k}{2}\cdot\frac{1}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Consequently
$$\frac{1}{a} = \left(\frac{2}{r} - \frac{v^2}{k}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Putting the initial values of v and r in this, we get

$$\frac{1}{a} = \frac{2}{d} - 2n\frac{k}{d}\cdot\frac{1}{k} = \frac{2}{d}(1 - n) \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Thus
$$2a = \frac{d}{1 - n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Now, by using two well-known properties of the ellipse, we can describe the orbit at once. These properties are

(1) The sum of the focal distances of any point on the ellipse is equal to the major axis $2a$.

(2) The focal distances of any point on the ellipse make equal angles with the tangent at that point.

Let I be the image of S in the tangent PQ. Join IP and produce to S′, so that IS′ = $2a$. Then S′ is clearly the other focus; for, by the construction,

$$SP + PS' = IP + PS' = 2a,$$

and the angle SPB is equal to the angle S′PQ.

Now that the two foci are known, and the length of the major axis, it is an easy matter to draw the ellipse.

It should be noticed that a, and therefore the period τ, depends only on d and n for a given attracting centre. Hence, if different bodies be sent off from the same point with the same velocities but in different directions, their major axes and their periods will all be the same, because n depends only on the magnitude of V, and not on its direction.

360. If n is greater than 1 the orbit is an hyperbola, and we get the major axis by changing the sign of the right-hand side of (4). To find S′ in this case, we have to produce IP backwards and again make IS′ equal to $2a$. Any tangent to an hyperbola passes between the foci and makes equal angles with the focal distances. Also the *difference* of the focal distances is equal to the major axis.

361. There are two necessary conditions for a circular orbit. These conditions are—(1) that the direction of projection must be perpendicular to SP; and (2) that PS′ must be equal to SP, that is, a must be equal to d. This last condition makes n equal to $\frac{1}{2}$, and therefore $V^2 = \frac{k}{d}$.

If $n = 1$ the orbit is a parabola, and $V^2 = 2\frac{k}{d}$, twice the square of the circular velocity, as we have remarked before (Art. 356).

362. Time occupied by a Body describing a Parabolic Orbit in travelling from one end of the Latus Rectum to the other.—The time occupied is the area cut off the parabola by the latus rectum divided by $\frac{1}{2}h$, the area swept out by the radius vector in unit time. Now for a parabolic orbit

$$\tfrac{1}{2}v^2 = \frac{1}{2}\cdot\frac{h^2}{p^2} = \frac{k}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and the equation of the parabola is

$$\frac{l}{p^2} = \frac{2}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

By comparing (1) and (2) we find

$$h^2 = lk \, . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The area cut off a parabola by the latus rectum can easily be found by integration. Its value is

$$A = \tfrac{2}{3}l^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Hence the required time is

$$\frac{\frac{2}{3}l^2}{\frac{1}{2}\sqrt{lk}} = \frac{4}{3\sqrt{k}}l^{\frac{3}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

It is worth while to compare (3) with the corresponding equation for a circular orbit. In a circular orbit of radius a

$$v^2 = \frac{k}{a} \, . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and therefore

$$h^2 = a^2v^2 = ak \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Thus the rate of describing area in a parabolic orbit is the same as in a circular orbit the radius of which is equal to the semi latus rectum of the parabola.

363. We will now work a few examples on the inverse square law.

EXAMPLE I.—*The earth is at perihelion very nearly at mid-winter for northern latitudes. If it were exactly at mid-winter the equinoxes would occur when the earth is at the ends of that latus rectum which passes through the sun. Taking the eccentricity of its orbit as* $\frac{1}{60}$, *find the intervals between the equinoxes on this assumption.*

Let A denote the smaller area cut off the ellipse by the latus rectum through the sun. Then the time occupied in describing the portion of the orbit bounding this area being denoted by t_1, we get, since the radius vector describes area at a constant rate,

$$\frac{t_1}{\text{one year}} = \frac{A}{\text{area of ellipse}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

that is,

$$t_1 = \frac{A}{\pi ab} \text{ of a year} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Now, since the eccentricity is small, the distance ea between the centre of the ellipse and the sun is very small, and consequently the area between the minor axis and the latus rectum may be regarded as a rectangle, whose sides are ea and $2b$. The area of the rectangle is $2eab$, and therefore

$$A = \tfrac{1}{2}\pi ab - 2eab \, . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Hence $$t_1 = \frac{\frac{1}{2}\pi ab - 2eab}{\pi ab}\text{ year}$$

$$= \tfrac{1}{2}\left(1 - \frac{4e}{\pi}\right)\text{ year} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This differs from half a year by 3·8 days. This would be the interval from the autumn to the spring equinox.

364. Taking the eccentricity of the orbit of Mercury to be 0·2, and its period of revolution as 88 days, find

(1) the ratio of the least to the greatest velocity;

(2) its major axis in terms of that of the earth's orbit;

(3) its velocity at one end of the minor axis, making use of the earth's velocity.

(1) Since pv is constant, the velocity varies inversely as p. The perpendicular p has its greatest and least values when the planet is at aphelion and perihelion respectively (the ends of the major axis). These values of p are $a + ea$ and $a - ea$. Hence the ratio of the least to the greatest velocity is

$$\frac{a - ea}{a + ea} = \frac{1 - e}{1 + e} = \frac{2}{3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

(2) If R is the radius of the earth's orbit, then

$$\frac{a^3}{R^3} = \left(\frac{88}{365\frac{1}{4}}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and therefore $$\frac{a}{R} = 0·387 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

(3) When the planet is at the end of the minor axis its distance from the sun is a. The energy equation gives

$$\tfrac{1}{2}v^2 = \frac{k}{r} - \frac{k}{2a}$$

$$= \frac{k}{a} - \frac{k}{2a} = \frac{1}{2}\cdot\frac{k}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Hence $$v = \sqrt{\frac{k}{a}} = \frac{1}{\sqrt{0·387}}\cdot\sqrt{\frac{k}{R}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Now the velocity in the earth's orbit (assumed circular) is

$$\sqrt{\frac{k}{R}} = \frac{2\pi \times 92·5 \times 10^6}{365\frac{1}{4} \times 24 \times 60 \times 60} = 18·5 \text{ miles per sec.} \quad . \quad (6)$$

Consequently

$$v = \frac{1}{\sqrt{0·387}} \times 18·5 = 29·7 \text{ miles per sec.} \quad . \quad . \quad . \quad (7)$$

365. *A body is projected from the earth's surface with a velocity* $n\sqrt{Rg}$, n *being a number and* R *the earth's radius: to find the major axis of the orbit relative to the earth, neglecting the attraction of other bodies.*

The attraction of the earth on unit mass at distance r is

$$g\frac{R^2}{r^2} \quad \ldots \ldots \ldots \ldots \quad (1)$$

Hence the energy equation of the projected body is

$$\tfrac{1}{2}v^2 = \frac{gR^2}{r} - \frac{gR^2}{2a} \quad \ldots \ldots \ldots \quad (2)$$

Since $v = n\sqrt{Rg}$ when $r = R$, this gives

$$n^2Rg = 2Rg - \frac{R^2g}{a} \quad \ldots \ldots \ldots \quad (3)$$

whence
$$a = \frac{R}{2 - n^2} \quad \ldots \ldots \ldots \quad (4)$$

If $n^2 = 2$ this gives $a = \infty$, and the orbit is therefore a parabola.

Since $\sqrt{Rg}$ is about 26,000 feet per second, n is usually a small quantity for any actual projectile at the earth's surface. Thus taking the velocity of projection of a shot from a large gun as 2000 feet per second, we find $n = \frac{1}{13}$. Hence

$$a = \frac{R}{2 - \frac{1}{169}} = \frac{R}{2}(1 + \tfrac{1}{338}) \text{ nearly} \quad \ldots \ldots \quad (5)$$

Thus the major axis of the orbit differs from the radius of the earth by $(2a - R) = \frac{1}{338}R = 11{\cdot}7$ miles about. If the projectile is started in a horizontal direction, the other end of the major axis is therefore 11·7 miles beyond the earth's centre.

366. Given the Orbit, to find the Law of Attraction.—Suppose the tangential-polar equation of the orbit is known; that is, p is known in terms of r.

The energy equation gives

$$\frac{1}{2}\cdot\frac{h^2}{p^2} = -\int P\,dr \quad \ldots \ldots \ldots \quad (1)$$

Differentiating with respect to r,

$$\frac{h^2}{p^3}\cdot\frac{dp}{dr} = P \quad \ldots \ldots \ldots \ldots \quad (2)$$

This gives P in terms of r, since the equation to the orbit enables us to express p in terms of r.

EXAMPLE 1.—*Conic about a focus.*

Here
$$\frac{l}{p^2} = \frac{2}{r} \pm \frac{1}{a} \quad \ldots \ldots \ldots \ldots \quad (3)$$

Differentiating both sides with respect to r,

$$-2\frac{l}{p^3}\cdot\frac{dp}{dr} = -\frac{2}{r^2} \quad \ldots \ldots \quad (4)$$

Therefore $$P = \frac{h^2}{l} \cdot \frac{1}{r^2} = \frac{k}{r^2} \quad . \quad . \quad . \quad . \quad (5)$$

If the orbit is the branch of an hyperbola farthest from the centre of force, the equation is

$$\frac{l}{p^2} = -\frac{2}{r} + \frac{1}{a} \quad . \quad . \quad . \quad . \quad (6)$$

Thus $$P = -\frac{h^2}{l} \cdot \frac{1}{r^2} \quad . \quad . \quad . \quad . \quad (7)$$

Since a positive value of P denotes an attraction, a negative value must denote a repulsion. Hence for this orbit the law of force differs from the gravitation law only in being a repulsion instead of an attraction.

EXAMPLE 2.—*Circle with pole on circumference.*

Here $$p = \frac{r^2}{2a} \quad . \quad . \quad . \quad . \quad (8)$$

Therefore $$\frac{1}{p^3} \cdot \frac{dp}{dr} = \left(\frac{2a}{r^2}\right)^3 \cdot \frac{r}{a} = \frac{8a^2}{r^5} \quad . \quad . \quad . \quad . \quad (9)$$

and consequently $$P = \frac{8h^2a^2}{r^5} \quad . \quad . \quad . \quad . \quad (10)$$

367. Orbits in Polar Co-ordinates.—If the pole is at the centre of force the equations of motion for unit mass are

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -P \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0 \quad . \quad . \quad . \quad . \quad (2)$$

and (2) gives $$r^2\frac{d\theta}{dt} = h \quad . \quad . \quad . \quad . \quad (3)$$

To get the differential equation of the orbit we must eliminate t from (1) and (3). This differential equation takes a simpler form if $\frac{1}{r}$ be taken as the dependent variable instead of r. Let, therefore,

$$u = \frac{1}{r} \quad . \quad . \quad . \quad . \quad (4)$$

Equation (3) gives

$$\frac{d\theta}{dt} = \frac{h}{r^2} = hu^2 \quad . \quad . \quad . \quad . \quad (5)$$

Then $$\frac{dr}{dt} = \frac{dr}{du} \cdot \frac{du}{d\theta} \cdot \frac{d\theta}{dt} = -\frac{1}{u^2} \cdot \frac{du}{d\theta} \cdot hu^2 = -h\frac{du}{d\theta} \quad . \quad . \quad . \quad . \quad (6)$$

and $$\frac{d^2r}{dt^2} = \frac{d}{dt}\left(-h\frac{du}{d\theta}\right) = \frac{d\theta}{dt} \cdot \frac{d}{d\theta}\left(-h\frac{du}{d\theta}\right) = -h^2u^2\frac{d^2u}{d\theta^2} \quad . \quad . \quad (7)$$

By means of (5) and (7), equation (1) can now be written

$$-h^2u^2\frac{d^2u}{d\theta^2} - h^2u^3 = -P \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

or

$$\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

This differential equation gives the orbit when P is known, and it gives P when the orbit is known.

368. A first integral of equation (9) in the last article can always be found if we can integrate P with respect to r.

Multiplying both sides of (9) by $\frac{du}{d\theta}$, we get

$$\frac{du}{d\theta}\cdot\frac{d^2u}{d\theta^2} + u\frac{du}{d\theta} = \frac{P}{h^2u^2}\cdot\frac{du}{d\theta} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Integrating this with respect to θ,

$$\tfrac{1}{2}\left\{\left(\frac{du}{d\theta}\right)^2 + u^2\right\} = \int\frac{P}{h^2u^2}\cdot\frac{du}{d\theta}d\theta + C$$

$$= \int\frac{P}{h^2u^2}du + C \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This is simply a disguised form of energy equation, for it corresponds to equation (6), Art. 350.

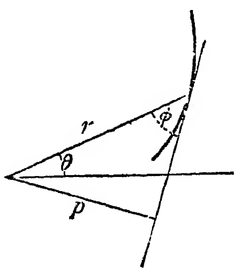


FIG. 170.

From Fig. 170

$$p = r\sin\phi$$

Therefore

$$\frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi$$

$$= \frac{1}{r^2}(1 + \cot^2\phi)$$

$$= u^2\left\{1 + \left(\frac{dr}{rd\theta}\right)^2\right\}$$

$$= u^2 + \left(\frac{du}{d\theta}\right)^2 \quad . \quad . \quad . \quad (3)$$

since

$$\frac{dr}{r} = -\frac{du}{u}$$

Thus, equation (2) can be written

$$\frac{1}{2}\cdot\frac{1}{p^2} = \int\frac{P}{h^2u^2}du + C = -\int\frac{P}{h^2}dr + C \quad . \quad . \quad . \quad (4)$$

which is the same equation as (6), Art. 350.

Since $\frac{h}{p}$ is the velocity, it follows from (3) that the square of the velocity is

$$h^2\left\{\left(\frac{du}{d\theta}\right)^2 + u^2\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

This expression can be easily obtained directly. For the radial and transverse velocities are $\frac{dr}{dt}$ and $r\frac{d\theta}{dt}$. Hence

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 = \left(\frac{1}{u^2}\cdot\frac{du}{dt}\right)^2 + \frac{1}{u^2}\cdot\left(\frac{d\theta}{dt}\right)^2 \quad . \quad (6)$$

But
$$\frac{d\theta}{dt} = hu^2 \quad \text{or} \quad dt = \frac{d\theta}{hu^2} \quad . \quad . \quad . \quad (7)$$

and therefore

$$v^2 = \left(\frac{1}{u^2}hu^2\frac{du}{d\theta}\right)^2 + \frac{1}{u^2}h^2u^4$$

$$= h^2\left\{\left(\frac{du}{d\theta}\right)^2 + u^2\right\} \quad . \quad . \quad . \quad (8)$$

Equation (3) gives the relation necessary for transforming tangential-polar equations into ordinary polar equations.

369. We will now return to equation (9) of Art. 367 and find the orbits for certain laws of force.

Inverse cube law.

Here
$$P = \frac{k}{r^3} = ku^3 \quad . \quad . \quad . \quad (1)$$

The equation giving the orbit is

$$\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2} = \frac{k}{h^2}u \quad . \quad . \quad . \quad (2)$$

or
$$\frac{d^2u}{d\theta^2} + \left(1 - \frac{k}{h^2}\right)u = 0 \quad . \quad . \quad . \quad (3)$$

If $\frac{k}{h^2} > 1$, the solution is (Art. 337)

$$u = Ae^{\sqrt{\left(\frac{k}{h^2}-1\right)}\theta} + Be^{-\sqrt{\left(\frac{k}{h^2}-1\right)}\theta} \quad . \quad . \quad . \quad (4)$$

And if $\frac{k}{h^2} < 1$, the solution is (Art. 292)

$$u = C\sin\left\{\sqrt{\left(1 - \frac{k}{h^2}\right)}\theta + \beta\right\} \quad . \quad . \quad . \quad (5)$$

Since the equations of equiangular spirals have the forms

$$r = ae^{\pm c\theta} \quad . \quad . \quad . \quad (6)$$

it is clear that, if A or B is zero, the orbit in (4) is one of these spirals.

If B and A have opposite signs, and B is greater than A, there is a particular value of θ which makes u zero, and therefore r infinite. Whereas if B is less than A, u becomes infinite, and therefore r zero, when θ becomes infinite. The general curve given by (4) has thus this property in common with the equiangular spiral, that it leads ultimately either to the centre of force or to infinity.

It is easy to see the character of the orbit determined by (5). Whatever be the initial circumstances of the motion, r will become infinite for the first value of θ which makes the right-hand side zero.

370. Inverse Square Law.

$$P = \frac{k}{r^2} = ku^2 \quad \ldots \quad (1)$$

Hence
$$\frac{d^2u}{d\theta^2} + u = \frac{k}{h^2} \quad \ldots \quad (2)$$

This is similar to equation (8), Art. 296, and its solution is

$$u = \frac{k}{h^2} + A\cos(\theta + \beta) \quad \ldots \quad (3)$$

Now the polar equation of a conic with eccentricity e and latus rectum $2l$ is (see Appendix)

$$u = \frac{1}{r} = \frac{1}{l}(1 \pm e\cos\theta) \quad \ldots \quad (4)$$

when the pole is taken at one focus and the major axis is the initial line. The upper or the lower sign in the bracket must be taken according as the direction of the initial line is towards, or away from, the point on the conic nearest to the pole.

By comparing (3) and (4) we see that (3) represents a conic in which

$$\frac{k}{h^2} = \frac{1}{l} = \frac{a}{b^2} \quad \ldots \quad (5)$$

$$A = \pm\frac{e}{l} \quad \ldots \quad (6)$$

and the initial line makes an angle β with the major axis. If we shift the initial line back by an angle β, we must put θ instead of the old $(\theta + \beta)$. Then the equation becomes

$$\frac{1}{r} = \frac{k}{h^2} + A\cos\theta \quad \ldots \quad (7)$$

The form of the conic depends on A. Thus the orbit is an ellipse, a parabola, or an hyperbola, according as the magnitude of A is less than, equal to, or greater than $\frac{k}{h^2}$. But in all cases the orbit is a conic. We have already obtained this result by means of the tangential-polar equation, but tangential-polar equations are rather unsatisfactory things. One never feels quite sure that the tangential-polar equation derived from one curve is not also the equation of some other curve. But the definiteness of the polar equation (7) removes all doubt whether the orbit can be anything but a conic.

371. Given the Orbit, to find the Law of Force.—The equation giving P is

$$P = h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

By means of the equation to the orbit $\frac{d^2u}{d\theta^2}$ can be expressed in terms of u, and therefore P can be expressed in terms of u, and that is the problem before us.

We will find the law of force for the system of curves.

$$r^n = a^n \cos n\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Taking logs of both sides, we get

$$n \log_e r = -n \log_e u = \log_e a^n + \log_e \cos n\theta \quad . \quad . \quad (3)$$

Differentiating with respect to θ,

$$-\frac{n}{u}\cdot\frac{du}{d\theta} = -n\frac{\sin n\theta}{\cos n\theta}$$

or

$$\frac{1}{u}\cdot\frac{du}{d\theta} = \tan n\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Differentiating again,

$$\frac{1}{u}\cdot\frac{d^2u}{d\theta^2} - \frac{1}{u^2}\left(\frac{du}{d\theta}\right)^2 = n \sec^2 n\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Therefore, putting $\tan n\theta$ for $\frac{1}{u}\cdot\frac{du}{d\theta}$ in (5),

$$\frac{1}{u}\cdot\frac{d^2u}{d\theta^2} = \tan^2 n\theta + n \sec^2 n\theta$$

$$= (n+1)\sec^2 n\theta - 1$$

$$= (n+1)\frac{a^{2n}}{r^{2n}} - 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Hence

$$P = h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) = \frac{h^2}{r^2}(n+1)\frac{a^{2n}}{r^{2n}}u$$

$$= (n+1)h^2a^{2n}\,\frac{1}{r^{2n+3}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Thus $P \propto \frac{1}{r^{2n+3}}$ for this curve.

Many well-known curves are included in the form (2), as is shown by the following list :—

$n = 1$ gives a circle with pole on circumference.
$n = -2$ gives a rectangular hyperbola with pole at centre.
$n = -\frac{1}{2}$ gives a parabola with pole at focus.
$n = -1$ gives a straight line.
$n = \frac{1}{2}$ gives a cardioid with pole at cusp.

Equation (7) shows that for the straight line ($n = -1$) P is zero. This, of course, must be the case if the straight line does not pass through the pole.

372. Let the equation of an orbit be

$$r = a\cos^2\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Then
$$u = \frac{1}{a}\sec^2\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\frac{d^2u}{d\theta^2} = \frac{1}{a}(6\sec^4\theta - 4\sec^2\theta)$$

$$= \frac{1}{a}(6a^2u^2 - 4au) \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Therefore
$$P = h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right)$$

$$= 3h^2\left(\frac{2a}{r^4} - \frac{1}{r^3}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

373. Time taken to describe an Arc of a Parabola by a body attracted towards the focus.—The equation to a parabola referred to focus as pole and the line towards the vertex as initial line is

$$\frac{l}{r} = 1 + \cos\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now, for a central force through the pole

$$r^2\frac{d\theta}{dt} = h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

that is,
$$\frac{dt}{d\theta} = \frac{r^2}{h}$$

$$= \frac{l^2}{h}\cdot\frac{1}{(1+\cos\theta)^2} \text{ by (1)}$$

$$= \frac{l^2}{h}\cdot\frac{1}{4\cos^4\frac{\theta}{2}}$$

$$= \frac{l^2}{4h}\sec^4\frac{\theta}{2}$$

$$= \frac{l^2}{4h}\left(\tan^2\frac{\theta}{2} + 1\right)\sec^2\frac{\theta}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Hence
$$t = \frac{l^2}{4h}\int\left(\tan^2\frac{\theta}{2} + 1\right)\sec^2\frac{\theta}{2}d\theta$$

$$= \frac{l^2}{2h}\left(\frac{1}{3}\tan^3\frac{\theta}{2} + \tan\frac{\theta}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

No constant need be added if the time from the vertex (or perihelion) is required.

Now by equation (6), Art. 356,

$$h^2 = kl. \quad \cdots\cdots \quad (5)$$

Also, if d is the perihelion distance,

$$l = 2d \quad \cdots\cdots \quad (6)$$

Hence, substituting for h and l in (4),

$$t = \sqrt{\frac{2d^3}{k}}\left(\tfrac{1}{3}\tan^2\frac{\theta}{2} + 1\right)\tan\frac{\theta}{2} \quad \cdots \quad (7)$$

It is often convenient to have t expressed in terms of r. From (1)

$$r = \frac{l}{1+\cos\theta} = \frac{l}{2\cos^2\frac{\theta}{2}} = d\sec^2\frac{\theta}{2} = d\left(1+\tan^2\frac{\theta}{2}\right). \quad (8)$$

Thus

$$\tan^2\frac{\theta}{2} = \frac{r-d}{d} \quad \cdots\cdots \quad (9)$$

Substituting in (7) for $\tan\frac{\theta}{2}$

$$t = \sqrt{\frac{2d^3}{k}}\left(\frac{r+2d}{3d}\right)\sqrt{\frac{r-d}{d}}$$

$$= \sqrt{\frac{2(r-d)}{k}}\cdot\frac{r+2d}{3} \quad \cdots\cdots \quad (10)$$

374. Time taken to describe an Arc of an Ellipse under a Force towards One Focus.—The time can be obtained by integrating $\frac{r^2}{h}$ with respect to θ, as was done for the parabolic orbit. But the integration is more troublesome in this case. We shall therefore use a different method.

Let S be the point of attraction, P the position of the particle at time t after passing A, the perihelion point. P′ is the point on the auxiliary circle corresponding to P. Since $\frac{1}{2}h$ is the area swept over by the radius vector in unit time, it follows that

$$t = \frac{2(\text{area of sector SAP})}{h} \quad (1)$$

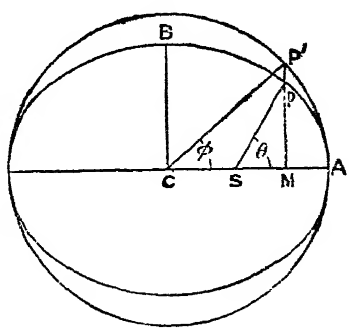


FIG. 171.

Now, if (x, y) are the co-ordinates of a point on the ellipse referred to rectangular axes CA, CB, and if ϕ denotes the eccentric angle of P,

$$\text{area MAP} = \int_{a\cos\phi}^{a} y\,dx \quad \cdots \quad (2)$$

But $x = a\cos\phi'$, $y = b\sin\phi'$, where ϕ' is the eccentric angle of the point (x, y). Then

$$dx = -a\sin\phi' d\phi' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Hence

$$\text{area MAP} = \int_0^\phi ab\sin^2\phi' d\phi'$$

$$= \tfrac{1}{2}ab\int_0^\phi (1 - \cos 2\phi')d\phi'$$

$$= \tfrac{1}{2}ab(\phi - \tfrac{1}{2}\sin 2\phi) \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Thus

$$\begin{aligned}\text{area SAP} &= \tfrac{1}{2}ab(\phi - \tfrac{1}{2}\sin 2\phi) + \tfrac{1}{2}\text{SM}\cdot\text{MP} \\ &= \tfrac{1}{2}ab(\phi - \sin\phi\cos\phi) + \tfrac{1}{2}(\text{CM} - \text{CS})\text{MP} \\ &= \tfrac{1}{2}ab(\phi - \sin\phi\cos\phi) + \tfrac{1}{2}(a\cos\phi - ae)b\sin\phi \\ &= \tfrac{1}{2}ab(\phi - e\sin\phi) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)\end{aligned}$$

Hence

$$t = \frac{ab}{h}(\phi - e\sin\phi) = \sqrt{\frac{a^3}{k}}(\phi - e\sin\phi) \quad . \quad . \quad (6)$$

since

$$h^2 = kl = k\frac{b^2}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

To express t in terms of θ we must find a relation between ϕ and θ. Thus, since MP and SP are the y and r of the point P,

$$b\sin\phi = y = r\sin\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

And by the equation of the ellipse

$$r = \frac{l}{1 + e\cos\theta} = \frac{a(1 - e^2)}{1 + e\cos\theta} \quad . \quad . \quad . \quad . \quad (9)$$

Therefore (8) and (9) give

$$\sin\phi = \frac{a}{b}\cdot\frac{(1 - e^2)\sin\theta}{(1 + e\cos\theta)} = \sqrt{1 - e^2}\frac{\sin\theta}{1 + e\cos\theta} \quad . \quad . \quad (10)$$

This gives the value of ϕ corresponding to any value of θ. If we put the value of ϕ in (6) and substitute for h from (7), we get

$$t = \frac{a^{\frac{3}{2}}}{\sqrt{k}}\left\{\sin^{-1}\left(\frac{\sqrt{1 - e^2}\sin\theta}{1 + e\cos\theta}\right) - \frac{e\sqrt{1 - e^2}\sin\theta}{1 + e\cos\theta}\right\} \quad . \quad (11)$$

When θ increases by a multiple of π, ϕ increases by the same amount. Putting $\theta = 2\pi$, we get the result already given for the whole period, namely,

$$t = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{k}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Writing τ for the whole period we may express the time for any value of θ thus—

$$t = \frac{\tau}{2\pi}\left(\sin^{-1}\frac{\sqrt{1 - e^2}\sin\theta}{1 + e\cos\theta} - \frac{e\sqrt{1 - e^2}\sin\theta}{1 + e\cos\theta}\right) \quad . \quad (13)$$

375. A few examples will now be worked in illustration of the formulæ.

EXAMPLE 1.—*If the perihelion distance of a comet, which describes a parabola about the sun, is equal to the earth's mean distance from the sun, to find the time it takes to travel* 60° *from perihelion.*

By equation (7), Art. 373,

$$t = \sqrt{\frac{2d^3}{k}}\left(\tfrac{1}{3}\tan^3 30° + \tan 30°\right)$$

$$= \sqrt{2}\sqrt{\frac{d^3}{k}}\left(\frac{10}{9\sqrt{3}}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

But since d is the earth's mean distance,

$$2\pi\sqrt{\frac{d^3}{k}} = 1 \text{ year} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Therefore $$t = \frac{10\sqrt{2}}{9\sqrt{3}\cdot 2\pi} = \frac{5\sqrt{6}}{27\pi} = 0{\cdot}150 \text{ year}$$

$$= 55 \text{ days nearly} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

EXAMPLE 2.—*If the perihelion distance of a comet describing a parabola is* 0·586 *of the earth's mean distance. to find how long it takes to describe* 80° *from perihelion and to find its distance from the sun then.*

Writing R for the earth's distance from the sun, we get—

$$t = \sqrt{\frac{2(0{\cdot}586)^3R^3}{k}}\left(\tfrac{1}{3}\tan^3 40° + \tan 40°\right)$$

$$= \frac{\sqrt{2(0{\cdot}586)^3}}{2\pi}(1{\cdot}036) \text{ year}$$

$$= 0{\cdot}1010 \times 1{\cdot}036 \text{ year} = 0{\cdot}1046 \text{ year}$$

$$= 38{\cdot}2 \text{ days} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

To find the distance from the sun we must use the equation of the orbit

$$\frac{l}{r} = 1 + \cos\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

When $\theta = 80°$ we get

$$r = \frac{l}{1 + \cos 80°} = \frac{2(0{\cdot}586)R}{1{\cdot}174} = R \text{ nearly} \quad . \quad . \quad . \quad (6)$$

EXAMPLE 3.—*The period of Halley's comet is observed to be about* 75 *years, and its perihelion distance is* 0·586R (*the same distance as we took for the parabolic orbit in the last example*), *to find the time of describing* 80° *from perihelion and the distance of the comet from the sun at that time.*

Since the eccentricity is not unity in this case the work is rather more laborious than for the parabolic orbit.

By comparing the period with that of the earth we get the ratio of a to R thus—

$$\left(\frac{a}{\mathrm{R}}\right)^{\frac{3}{2}} = \frac{75}{1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Hence
$$\frac{a}{\mathrm{R}} = (75)^{\frac{2}{3}} = 17\cdot78 \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Now the perihelion distance is $a(1 - e)$. Consequently

$$a(1 - e) = 0\cdot586\mathrm{R}$$

or
$$e = -0\cdot586\frac{\mathrm{R}}{a} + 1 = 0\cdot96704 \quad . \quad . \quad . \quad (9)$$

Therefore
$$\sqrt{1 - e^2} = \sqrt{0\cdot06484} = 0\cdot2546 \quad . \quad . \quad . \quad . \quad . \quad (10)$$

When $\theta = 80°$

$$r = \frac{l}{1 + e\cos 80°} = \frac{a(1 - e^2)}{1 + (0\cdot967)(0\cdot1736)} = 0\cdot987\mathrm{R} \quad . \quad (11)$$

and by equation (13), Art. 374,

$$t = \frac{75}{2\pi}\left\{\sin^{-1}\frac{0\cdot2546 \sin 80°}{1 + (0\cdot967)(0\cdot1736)} - \frac{(0\cdot967)(0\cdot2546)\sin 80°}{1 + (0\cdot967)(0\cdot1736)}\right\}$$

$$= \frac{75}{2\pi}\{\sin^{-1}(0\cdot2147) - (0\cdot9670)(0\cdot2147)\}$$

$$= \frac{75}{2\pi}(0\cdot00875) = 0\cdot1044 \text{ year}$$

$$= 38\cdot1 \text{ days} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

376. The close agreement between the results in the last two examples shows how difficult it is to distinguish between a parabolic orbit and a long elliptic one by means of observations made near a focus. Since a comet is visible to us only while it is in the neighbourhood of the earth's orbit, it is easy to see then why the path of a comet is always assumed to be a parabola until very exact observations have proved that it is not one.

377. A Circular Orbit is always possible under a Central Force.—When a particle is acted on by a force towards a fixed point which is a function only of the distance from that point a circular orbit is always possible, the centre of the circle being at the point of attraction. For, if the body be started at distance a from the centre at right angles to the radius vector through that point with a velocity given by

$$\frac{v^2}{a} = f(a) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

(assuming that $f(r)$ denotes the attraction on unit mass at distance r) all the conditions for circular motion are satisfied, and consequently the orbit will be a circle of radius a.

That a circular orbit is always possible can easily be shown from the equations of motion. The equations are

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -P$$

$$= -f(r) \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and

$$\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

These equations are clearly satisfied by $r = a$, $\frac{d\theta}{dt} = n$, where a and n are constants, provided only

$$an^2 = f(a) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

which is the same condition as (1), because $v = an$ when a is constant.

The radius a may have any magnitude whatever provided v is determined from (1).

378. Stability in a Circular Orbit.—But although a circle is a possible orbit when the initial conditions are exactly adjusted, it may happen that a slight deviation from these conditions will cause the particle to wander, after a lapse of time, far from the original circle. Or, on the other hand, a slight deviation may cause the particle to describe an orbit every point of which is but slightly removed from the circular one. In the former case the circular orbit is said to be unstable, and in the latter case, stable. Our aim at present is to establish a criterion for determining whether, for any given law of force, a circular orbit is stable or unstable. In those cases where the equation to the orbit can be calculated for any possible initial conditions this criterion is, of course, unnecessary. For instance, in the case of the inverse square law of attraction we know that the orbit is a conic with the pole at one focus. It is, therefore, quite obvious, from considerations of continuity, that a body, starting under conditions differing only slightly from those necessary for circular motion, will describe a nearly circular ellipse, for this is the curve which comes next to the circle in the series of possible conics. But we are unable to calculate the general orbit except for a few laws of force, and it is for the general case that our criterion is useful.

379. Let us suppose that a particle is moving at one instant under conditions which very nearly agree with those necessary for circular motion, and that ever afterwards the particle is subject to no forces but the central one. Since P, the attraction on unit mass, is a function of u, we may write it in the form

$$P = ku^2 f(u) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now the differential equation of the orbit is

$$\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2} = \frac{k}{h^2}f(u) \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where h denotes, as usual, the constant moment of the velocity about the centre of attraction.

Now let a be a constant determined by the equation

$$a = \frac{k}{h^2}f(a) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The meaning of a is easily deduced from the fact that $u = a$ satisfies equation (2). Thus $\frac{1}{a}$ is the radius of the circular orbit which a particle could describe if it had the same h as the particle we are dealing with.

Since the central orbit is not exactly circular, let us put

$$u = a + x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where x is at first small. We have now to discover under what conditions x will for ever remain small.

Putting $a + x$ for u in (2), we get

$$\frac{d^2x}{d\theta^2} + a + x = \frac{k}{h^2}f(a + x) \quad . \quad . \quad . \quad . \quad . \quad (5)$$

As long as x is small we can expand the right-hand side of (5) by Taylor's theorem and consider only the first power of x. Thus

$$\frac{d^2x}{d\theta^2} + a + x = \frac{k}{h^2}\{f(a) + xf'(a)\} \quad . \quad . \quad . \quad . \quad (6)$$

Since a satisfies (3), this last equation gives

$$\frac{d^2x}{d\theta^2} + x = \frac{k}{h^2}xf'(a) = \frac{a}{f(a)}xf'(a) \quad . \quad . \quad . \quad . \quad (7)$$

or

$$\frac{d^2x}{d\theta^2} + x\left\{1 - a\frac{f'(a)}{f(a)}\right\} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If the coefficient of x is positive in (8), this takes the familiar form (Art. 292)

$$\frac{d^2x}{d\theta^2} + c^2x = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

the solution of which is

$$x = A\cos(c\theta + \beta) \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Hence, if A is small, as it generally would be with the initial conditions we have assumed, x is always small, and the orbit is therefore nearly circular. In this case the circular orbit is stable.

Since
$$r = \frac{1}{u} = \frac{1}{a + x} = \frac{1}{a} - \frac{x}{a^2} \text{ nearly} \quad . \quad . \quad . \quad . \quad (11)$$

we see that the motion may be regarded as a motion along the circle $r = \frac{1}{a}$, together with small oscillations in and out of this circle.

But if the coefficient of x in (8) is positive, the equation has the form

$$\frac{d^2x}{d\theta^2} - b^2x = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

the solution of which is, by Art. 337,

$$x = Ce^{b\theta} + De^{-b\theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and thus x is big when θ is big (unless the initial conditions are so nicely adjusted that C is exactly zero) until the approximate equation (8) ceases to be valid.

The condition, therefore, that a circular orbit should be stable is that

$$1 - a\frac{f'(a)}{f(a)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

should be positive, where $\frac{1}{a}$ denotes the radius of the circle. Whereas if this quantity is negative the circular orbit is generally unstable, and if it is zero the above investigation gives us no information concerning stability. We should have to take account of higher powers of x in (8) to decide the question in the last case.

380. Stability for Particular Laws of Force.—If the attraction varies inversely as the nth power of the distance, then—

$$u^2 f(u) = u^n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

that is,

$$f(a) = a^{n-2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence

$$1 - \frac{af'(a)}{f(a)} = 1 - (n-2) = 3 - n \quad . \quad . \quad . \quad . \quad (3)$$

Thus a circular orbit with any radius is stable if n is less than 3. When $n = 3$ the method of the last article fails, but in this case we can solve the differential equation exactly. Here

$$f(u) = u \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

On putting $(a + x)$ for u in the differential equation, we get

$$\frac{d^2u}{d\theta^2} + a + x = \frac{k}{h^2}(a + x) \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Now, since h has to be the same as for a circular orbit of radius $\frac{1}{a}$, equation (3) of the last article must be satisfied. That is,

$$a = \frac{k}{h^2}a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This equation does not determine a as it does for every other law of force, but it gives h in terms of k. Putting 1 for $\frac{k}{h^2}$ in (5), we get

$$\frac{d^2u}{d\theta^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Therefore

$$u = A\theta + B = A(\theta + \alpha) \quad . \quad . \quad . \quad . \quad . \quad (8)$$

This is a sort of spiral leading either towards the pole or towards infinity. Thus a circular orbit is unstable. The equation (8) ceases to hold, of course, when $A\theta$ is large.

For the inverse square law of attraction

$$f(a) = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and therefore $$1 - a\frac{f'(a)}{f(a)} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Hence $$x = A\cos(\theta + \beta) \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

that is, if A is small, $$r = \frac{1}{a} - \frac{1}{a^2}A\cos(\theta + \beta) \quad . \quad . \quad . \quad . \quad (12)$$

By comparing with previous results we know that $\frac{1}{a}$ is l, the semi latus rectum. For example, the earth's orbit, which is an ellipse with eccentricity $\frac{1}{60}$, can be very accurately represented by

$$r = l(1 - \tfrac{1}{60}\cos\theta) \quad . \quad . \quad . \quad . \quad . \quad (13)$$

θ being measured from perihelion.

This gives the correct value of r when $\theta = 90°$, but leaves slight errors at perihelion and aphelion. If we want the equation to give correct values of r at perihelion and aphelion we must replace l by a_1, the semi-major axis. Then

$$r = a_1(1 - \tfrac{1}{60}\cos\theta) \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

381. If the velocity of a particle, acted on by any system of conservative forces, be suddenly reversed, the particle will retrace its orbit in the backward direction.

We are supposing that the particle is moving in one direction with velocity v, and that its velocity is suddenly changed into v along the same line, but in the opposite direction.

Since the forces are conservative they have a potential. Let V denote the potential at any point.

To prove that the particle will retrace its path let us suppose that it is constrained to do so, and we shall show that there is no pressure on the constraining curve. We are to suppose that there is some arrangement, such as a smooth tube or wire, along the curve described by the particle in the free forward motion, so that when its velocity is reversed the particle is obliged to move along the same curve.

The energy equation tells us that

$$\tfrac{1}{2}mv^2 + V = \text{const.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

And since, immediately after the reversal of the velocity, the energy is the same as before, it follows that the velocity at any point in the backward motion is the same as it was in the forward motion at the same point. Also, if N is the whole normal force on the particle and ρ the radius of curvature,

$$m\frac{v^2}{\rho} = N \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Since v and ρ are just the same as in the forward motion, N is therefore the same. But the normal force in the backward motion is the

same as the normal force in the forward motion together with the pressure of the constraining curve. Hence the pressure of the curve is zero. That is, the particle will retrace its path without a constraining curve.

382. A point on a central orbit at which the tangent is perpendicular to the radius vector from the centre of force is called an *apse.* The corresponding distance from the centre is an *apsidal distance.* The angle between the radii to two successive apsides is an *apsidal angle.*

We shall now prove that in any central orbit there are but two different apsidal distances and one apsidal angle.

If the direction of motion of a particle were suddenly changed at an apse, the speed being the same as before, the particle would retrace its path. But since the direction of motion at an apse is perpendicular to the radius vector, it is clear that the orbit described in the backward motion would be the image in the apsidal distance of the orbit described in the forward motion. That is, the real orbit is symmetrical about any apsidal distance. Hence the two apsidal angles on each side of any apse are equal, and likewise the two apsidal distances on each side are equal. The apsidal distance of the apse we are considering is equal to the next but one on either side. Thus all apsidal angles are equal, and alternate apsidal distances are equal.

The apsidal distances can be found by putting r for p in the tangential polar equation (6) of Art. 350; or by putting zero for $\frac{du}{d\theta}$ in equation (2) in Art. 368. When the orbit is a conic about a focus the apsidal angle is 180° and the apsidal distances are the two portions of the major axis, namely $a(1 - e)$ and $a(1 + e)$.

In elliptic harmonic motion, where the centre of the ellipse is the point of attraction, the apsidal distances are the semi-axes and the apsidal angle is 90°.

When the force varies inversely as the cube of r, there is not more than one apse whatever be the character of the orbit.

383. Apsidal Angle in a nearly Circular Orbit.—With the same value of c as in equation (9), Art. 379, the radius vector in a nearly circular orbit is given by the equation

$$\frac{1}{r} = u = a + A\cos(c\theta + \beta) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

At an apse r is either a maximum or a minimum, and therefore u is either a minimum or a maximum. At these points it is obvious that

$$\cos(c\theta + \beta) = \pm 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

or

$$c\theta + \beta = n\pi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where n is any integer.

The apsidal angle is the difference between two successive values of θ given by (3). If ϵ denotes this angle we find

$$\epsilon = \frac{\pi}{c} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

For example, if the attraction is $\frac{k}{r^n}$ equation (3) of Art. 380 gives

$$c^2 = 3 - n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Hence the apsidal angle is

$$\epsilon = \frac{\pi}{\sqrt{3 - n}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Thus, as we already know, for the inverse square law, when $n = 2$

$$\epsilon = \pi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and for an attraction proportional to the direct distance, when $n = -1$,

$$\epsilon = \frac{\pi}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Suppose
$$P = k\left(\frac{1}{r^2} + \frac{b}{r^3}\right) = k(u^2 + bu^3) \quad . \quad . \quad . \quad . \quad (9)$$

in a nearly circular orbit. Here

$$f(u) = \frac{1}{u^2}(u^2 + bu^3) = 1 + bu \quad . \quad . \quad . \quad . \quad (10)$$

$$f'(u) = b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and therefore
$$1 - \frac{af'(a)}{f(a)} = \frac{1}{1 + ab} \quad . \quad . \quad . \quad . \quad (12)$$

Hence the apsidal angle is

$$\epsilon = \pi\sqrt{1 + ab} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

In this case the apsidal angle depends on a as well as b. That is, if b is fixed, the apsidal angle is different for different radii.

If $b = \frac{9}{16} \cdot \frac{1}{a}$, then

$$\epsilon = \tfrac{5}{4}\pi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

If $b = \frac{3}{a}$.

$$\epsilon = 2\pi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

384. *A particle of mass* M *is attached to a light string, which passes through a small hole in a smooth horizontal table on which the particle lies. The lower end of the string is fastened to one end of a flexible chain of mass* m *per unit length, which lies on the floor just underneath the hole in the table. The lower end of the string is held firmly at the level of the floor, while* M *is made to move in a circle on the table. If the string is let go while* M *is moving, to show that the particle always lies between two fixed circles in the subsequent motion.*

Let u be the radial component of the velocity of M at any instant, and v the transverse component. Then the upward velocity of the chain is clearly u. Let b be the radius of the circle which M was describing at the beginning, and r the distance of M from the hole at any instant.

The length of chain which has been raised off the floor is $(r-b)$, the increase in the radius vector to M. The centre of gravity of this is at a height $\frac{1}{2}(r-b)$, and therefore the work done in raising it is

$$\tfrac{1}{2}(r-b) \,.\, mg(r-b) = \tfrac{1}{2}mg(r-b)^2 \quad . \quad . \quad . \quad . \quad (1)$$

The work done by the weight is the negative of this.

The kinetic energy of the raised chain is

$$\tfrac{1}{2}m(r-b)u^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Equating the gain in kinetic energy of the particle and the chain to the work done on these bodies, we get

$$\tfrac{1}{2}\mathrm{M}(u^2+v^2) + \tfrac{1}{2}m(r-b)u^2 - \tfrac{1}{2}\mathrm{M}v_0^2 = -\tfrac{1}{2}mg(r-b)^2 \quad (3)$$

where v_0 was the velocity of M in its circle.

The tension in the connecting string clearly does no work on the system, for it does positive work in moving the chain up, and an equal amount of negative work on the particle as it moves outwards.

Since the force on M acts through a fixed point, the moment of momentum about this point is constant. Hence

$$rv = bv_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Substituting for v in (3) and rearranging the terms, we get

$$\{\mathrm{M} + m(r-b)\}u^2 = \mathrm{M}v_0^2\left(1 - \frac{b^2}{r^2}\right) - mg(r-b)^2$$

$$= \mathrm{M}v_0^2\frac{r^2-b^2}{r^2} - mg(r-b)^2$$

$$= \left\{\mathrm{M}v_0^2\frac{r+b}{r^2} - mg(r-b)\right\}(r-b) \quad . \quad (5)$$

If ever the whole of the chain lies on the floor, the term $m(r-b)u^2$ must be omitted from the left-hand side of (5), for this term arises from the kinetic energy of the chain, and this is clearly zero when all the chain is on the floor.

The left-hand side of (5) can never be negative, and consequently the right-hand side must not be negative. Now, when r is less than b, the factor $(r-b)$ on the right-hand side of (5) is negative, and the other factor is clearly positive for positive values of r. Hence r cannot be less than b, for that would make the left-hand side negative. We have now to show that there is an upper limit to r. When r is greater than b the factor $(r-b)$ is positive, and of the two terms in the other factor, namely,

$$\mathrm{M}v_0^2\frac{r+b}{r^2} \text{ which} = \mathrm{M}v_0^2\left(\frac{1}{r} + \frac{b}{r^2}\right) \quad . \quad . \quad . \quad (6)$$

and

$$mg(r-b) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

the first clearly decreases continually as r increases, and the second increases continually. Since the second one (7) starts at zero when $r=b$, it is clear that there is a particular value of r which makes the

two expressions (6) and (7) equal. After this point (7) goes on increasing and (6) decreasing as r increases, and consequently their difference, the first factor on the right-hand side of (5), is negative. Hence r cannot be greater than the single positive root of the equation

$$Mv_0^2\frac{r+b}{r^2} - mg(r-b) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Thus M is confined between two circles, the radius of the smaller being b, and that of the larger being the positive root of (8).

The radius of the larger circle depends on the value of

$$\frac{Mv_0^2}{mg} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Let this quantity be $\frac{25}{36}b^2$ in a particular case. Then the equation (8) becomes

$$\tfrac{25}{36}b^2\cdot\frac{r+b}{r^2} - (r-b) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

This is satisfied by $r = \frac{5}{3}b$.

To solve equation (8) in the general case it will usually be found easiest to plot the two curves

$$\left.\begin{aligned} y &= x - b \\ \text{and} \qquad y &= \frac{Mv_0^2}{mg}\cdot\frac{x+b}{x^2} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The value of x at the point of intersection is the value of r required.

The tension in the string when the particle was describing the circle of radius b was

$$T_0 = \frac{Mv_0^2}{b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

In the particular example worked out

$$\frac{Mv_0^2}{b} = \tfrac{25}{36}mbg \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

that is, the tension of the string just before it was released was equal to the weight of a length $\frac{25}{36}b$ of the chain.

385. Energy Equation for a Particle attracted to several Centres of Force.—The general equation of energy for a particle may be expressed thus—

$$\tfrac{1}{2}mv^2 - \tfrac{1}{2}mv_0^2 = W \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where W is the work done on the particle while its velocity changes from v_0 to v.

If the particle is attracted towards a fixed point with a force P, the work done in any displacement is

$$-\int_{r_0}^{r_1} P\,dr \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where r_0 and r_1 are the distances of the particle from the centre of attraction at the beginning and end of the displacement. If forces act

The line joining the particles makes angles with the axes, whose cosines are

$$\frac{x_2 - x_1}{\rho} \text{ and } \frac{y_2 - y_1}{\rho} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The equation of motion of m_1 parallel to OX is

$$m_1\frac{d^2x_1}{dt^2} = \frac{x_2 - x_1}{\rho}F \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The corresponding equation for m_2 is

$$m_2\frac{d^2x_2}{dt^2} = \frac{x_1 - x_2}{\rho}F \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Adding (2) and (3)

$$m_1\frac{d^2x_1}{dt^2} + m_2\frac{d^2x_2}{dt^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

But if (x, y) are the co-ordinates of the centre of mass of m_1, m_2, we have always

$$m_1x_1 + m_2x_2 = (m_1 + m_2)x \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Differentiating twice

$$m_1\frac{d^2x_1}{dt^2} + m_2\frac{d^2x_2}{dt^2} = (m_1 + m_2)\frac{d^2x}{dt^2} \quad . \quad . \quad . \quad . \quad (6)$$

Equations (4) and (6) show that

$$\frac{d^2x}{dt^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

In the same way we can prove that

$$\frac{d^2y}{dt^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Thus the centre of mass has no acceleration, and consequently this point has a constant velocity or is at rest. In either case we may regard the centre of mass as a fixed point in investigating the motion of each body, because accelerations relative to a point which has no acceleration are true accelerations, and the equations of motion involve only accelerations. The motion of the bodies relative to the centre of mass is therefore exactly the same, whether that point is at rest or moving with constant velocity. We shall therefore treat the centre of mass as a fixed point.

It should be noticed that the results just proved do not depend on the magnitude of F. They are true for all values of F provided only the force on one particle is equal and opposite to the force on the other. Moreover, if instead of only two particles there had been any number of particles, each acting on the others and being acted on by them, we should have found, on adding all the equations for motion parallel to OX, that the forces would have disappeared from the right-hand side

exactly as for two particles, and consequently the acceleration of the centre of mass of the system would have been zero just the same.

389. If two masses M, m, are at a distance ρ apart, the gravitation attraction between them is

$$F = \kappa \frac{Mm}{\rho^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let G be the position of the centre of mass, and let R, r, be the distances of the particles from G. Then, from the property of centre of mass

$$RM = rm \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Also

$$R + r = \rho \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From (2) and (3)

$$R = \frac{m}{M+m}\rho, \qquad r = \frac{M}{M+m}\rho \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Regarding G as a fixed point, the attraction on m is a force towards this fixed point, the magnitude of which force is

$$\kappa\frac{Mm}{\rho^2} = \kappa\frac{Mm}{r^2}\cdot\frac{M^2}{(M+m)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Thus the force acts towards the fixed point G, and varies inversely as the square of the distance from that point. Hence the orbit is a conic with G in one focus.

The same is true for the other mass M. But this can also be seen from the known orbit of m. For M and m are always on a line through G, and

$$\frac{R}{r} = \frac{m}{M}, \text{ a constant} \quad (6)$$

which shows that M describes an orbit similar to the one described by m with the point G as centre of similitude for the two orbits. Fig. 172 shows how the two orbits are related.

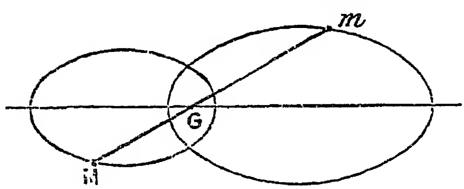


FIG. 172.

Since the dimensions of M's orbit are to those of m's in the ratio m : M, it is clear that, for cases such as the earth and the sun, where

$$\frac{\text{earth's mass}}{\text{sun's mass}} = \frac{1}{325000} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

we may safely regard the sun as a fixed point. It is on account of the smallness of the sun's orbit in comparison with the earth's that we say the earth goes round the sun, whereas, to be quite accurate, the sun and earth revolve round their common centre of mass—except in so far as they are disturbed by other bodies.

390. It is simpler to treat the motion of two bodies as a question of relative motion by regarding one of the bodies as fixed. Regarding M

as fixed, we find that the particle m describes an orbit relative to M exactly similar to its orbit relative to G. For

$$\frac{\rho}{r} = \frac{M + m}{M}, \text{ a constant} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and the direction of ρ in space is the same as that of r. Consequently the motion of m relative to M is an ellipse with M in one focus, and the size of this relative orbit is greater than the size of the orbit relative to G in the ratio $(M + m) : M$.

391. The Relative Orbit.—Assuming G to be fixed, the equations of motion of m are

$$\frac{d^2r}{dt^2} - r\overline{\frac{d\theta}{dt}}\Big|^2 = \kappa\frac{M}{\rho^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

On putting for r its value $\dfrac{M}{M + m}\rho$, these equations become

$$\frac{d^2\rho}{dt^2} - \rho\overline{\frac{d\theta}{dt}}\Big|^2 = \frac{M + m}{M} \cdot \kappa\frac{M}{\rho^2}$$

$$= \kappa\frac{M + m}{\rho^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\frac{d}{dt}\left(\rho^2\frac{d\theta}{dt}\right) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The equations (3) and (4) have exactly the same form as the equations of motion of a particle attracted towards a *fixed* particle of mass $(M + m)$. Thus the orbit of m relative to M is the same as if the attracting particle were fixed, and its mass were the sum of the masses.

This is a very useful result, because observations are always made on relative orbits. It would not be an easy matter, for example, to measure the distance of the moon from the centre of mass of earth and moon, but there is no extraordinary difficulty in measuring its distance from the centre of the earth.

In the last chapter it was shown that when the attraction towards a fixed centre is $\dfrac{k}{r^2}$ on unit mass, the period in an elliptic orbit is (Art. 357)

$$\tau = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{k}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

In this article $\kappa(M + m)$ corresponds to k of these earlier articles. Hence, if $2a$ is the major axis of the relative orbit, the period is

$$\tau = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{\kappa(M + m)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

κ being, in this case, the constant of gravitation.

392. Kepler's Laws.—Before any theory of gravitation was known, and while yet astronomy was mingled with the absurd pretensions of astrology, John Kepler (born 1571, died 1630) deduced, from a very extensive series of observations on the motions of the planets, but chiefly of the planet Mars, the three fundamental laws of planetary motion. These deductions are all the more wonderful because Kepler himself was not entirely free from the charlatanry of the astrologer. And besides, he had some preconceived fantastic notions of his own concerning the motions and distances of the planets, and these must have hampered him considerably in trying to find the laws of planetary motion. Yet, in spite of all these drawbacks, he managed to raise astronomy to the rank of a science, and to prepare the way for Newton's discovery of the law of gravitation.

Kepler's three laws are—

(1) *Every planet moves in an ellipse with the sun in one focus.*

(2) *The line joining a planet to the sun sweeps out equal areas in equal intervals of time.*

(3) *If the square of the period of each planet be divided by the cube of the major semi-axis of its orbit, the result is the same for every planet.*

The first law enables us to prove that the attraction on any planet varies inversely as the square of the distance of the planet from the sun.

The second law is a simple statement of the fact represented in mathematical symbols by the equation

$$\tfrac{1}{2}r^2\frac{d\theta}{dt} = \tfrac{1}{2}h, \text{ a constant} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

But we know from mechanical principles that

$$\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = \text{the moment of the forces about the origin.}$$

Hence Kepler's second law tells us that the forces which keep the planets in their orbits have no moments about the sun's centre, and therefore that they pass through that point. If observations were accurate enough to make this law indubitable, then the question whether the gravitation force is transmitted instantaneously would be settled.

The third law leads to a proof that the attraction of the sun on unit mass of any planet would be the same at the same distance if we assume what is proved by the first law, namely, that the force varies inversely as the square of the distance. It is a statement of the relation

$$\tau = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{\kappa(\mathrm{M} + m)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

provided we neglect the mass of the planet in comparison with that of the sun. If M is the sun's mass we get, on squaring both sides of (2) and neglecting m,

$$\tau^2 = \frac{4\pi^2}{\kappa\mathrm{M}}a^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The result has been obtained by assuming the inverse square law of attraction towards the sun, and this necessarily follows from Kepler's first and second laws The third law now tells us that the coefficient of a^3 in (3) is the same for all the planets. Thus κ is the same constant for all the planets.

393. Comparison of Masses of Bodies with Satellites.—The conclusions drawn from the hypothesis of universal gravitation have never been found to disagree with the results of observation. Thus the attraction between the earth and the moon is subject to the same laws as that between the sun and the planets. Moreover, Kepler's laws are true for the several satellites of a planet such as Jupiter or Saturn, provided the words "sun" and "planets" in Kepler's laws are replaced by "planet" and "satellites" respectively.

The gravitation theory, combined with observations on periods of revolution and astronomical distances, gives us a means of comparing the mass of a planet which has a satellite with the mass of the sun.

Let M and m denote the masses of the sun and any planet, τ the period of revolution of the planet, and a the major semi-axis of the relative orbit. Let M′ and m' denote the masses of a planet and its satellite, τ' the period of revolution of the satellite, and a' its mean distance from the planet. (By "mean distance" in astronomy the major semi-axis of the relative orbit is meant.)

Then
$$\tau^2 = \frac{4\pi^2 a^3}{\kappa(M+m)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\tau'^2 = \frac{4\pi^2 a'^3}{\kappa(M'+m')} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From these equations

$$\frac{M'+m'}{M+m} = \left(\frac{a'}{a}\right)^3 \cdot \left(\frac{\tau}{\tau'}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In every case m is negligible compared with M, because the mass of the largest planet Jupiter is only about $\frac{1}{1050}$ of the sun's mass. But there is sometimes good reason for retaining m', since, in the case of the moon, for instance, its mass is $\frac{1}{80}$ of the earth's mass (M′).

Now, although it is a difficult matter to reduce astronomical distances to terrestrial units, yet it is comparatively easy to find the ratios of astronomical distances by observation. Consequently the ratio $\frac{a'}{a}$ is known with considerable accuracy in most cases. The distance of the moon from the earth is an important exception. It is easier to express this distance in terrestrial units than in astronomical units. The quantity $\frac{\tau}{\tau'}$ can be determined to almost any required degree of accuracy. Hence equation (3) gives a very reliable value of the ratio of the masses.

There is no reason why the planet m should not be the same as M′, but any other planet will serve as well.

EXAMPLE.—*The period of Jupiter's revolution about the sun is* 11·86 *years, and the period of revolution of his fourth satellite is* 16·75 *days. Also the distance between the planet and this satellite subtends an angle* 8′ 25″ *at the sun's centre. To find the ratio of Jupiter's mass to the sun's mass.*

Here $$\frac{a'}{a} = \sin 8'25'' = 0{\cdot}002446$$

$$\frac{\tau}{\tau'} = \frac{11{\cdot}86 \times 365\frac{1}{4}}{16{\cdot}75} = 258{\cdot}6$$

Hence $$\frac{M' + m'}{M + M'} = (0{\cdot}002446)^3 \times (258{\cdot}6)^2$$

$$= \frac{1}{1021} \text{ nearly}$$

This differs from the accepted value by about 3 per cent. A decrease of 1 per cent. in $\frac{a'}{a}$ would give very nearly the accepted value.

394. The Hodograph of a Planet's Motion is a Circle.—Neglecting the action of all bodies except the sun, the path of a planet is an ellipse with the centre of mass of sun and planet in one focus.

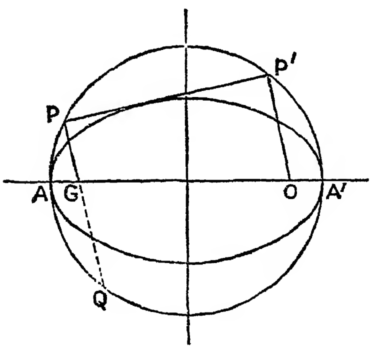


FIG. 173.

Let G be centre of mass of sun and planet, O the other focus of the ellipse. Let v be the velocity of the planet at any instant, p, p', the perpendiculars from G, O, on the line of motion, and P, P′, the feet of these perpendiculars.

Now by a well-known property of the ellipse, P and P′ lie on the auxiliary circle. But because the force always passes through G

$$pv = h \quad . \quad . \quad (1)$$

Now it is easy to show that if PG is produced to meet the circle again in Q

$$GQ = OP' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Therefore $$pp' = GP \,.\, GQ = AG \cdot GA' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which is constant for all positions of the particle.

Equations (1) and (3) show that v is proportional to p'. Therefore, by a suitable choice of scale we can make p' represent v in magnitude. Thus the vector OP′ would represent the velocity in magnitude and direction if it were turned through a right angle. The locus of P′ is therefore the hodograph of the motion turned through a right angle about O. But the locus of P′ is the auxiliary circle. Hence the

hodograph is obtained by turning the auxiliary circle through a right angle about O in that direction which brings $\overrightarrow{OP'}$ parallel to the velocity

We may take the auxiliary circle itself as the hodograph, but in this case the pole will be on the diameter perpendicular to AA′, and at a distance OA′ $(= a - ea)$ from one end.

There is a very interesting application of the above result to the theory of the aberration o[illegible] to this theory a fixed star at the pole of the ecliptic has, in consequence of the earth's motion, an apparent displacement which is proportional to the earth's velocity and in the same direction as that velocity. Such a star apparently moves, therefore, along a curve similar to the hodograph of the earth's motion. Thus the apparent path is a circle, and the true position of the star is not at the centre, but on the diameter parallel to the minor axis of the earth's orbit.

395.—A projectile thrown up from the earth's surface is subject to the same law of attraction as the moon. A projectile is therefore a small satellite to the earth while it is in motion. If the effect of the resistance of the air is neglected, the path of a projectile is a portion of a narrow ellipse with one focus at the earth's centre and the other somewhere near the point of projection. By assuming that the force acting on a particle in different positions acts in a fixed direction, instead of through the earth's centre, we are led to the conclusion that the path is a parabola. Now the portion of this parabola near the focus differs exceedingly little from the corresponding portion of the ellipse which is the true path. One end of a narrow ellipse (that is, an ellipse in which the minor axis is much smaller than the major axis) is very similar to a parabola whose focus coincides with that of the ellipse at the end considered. The eccentricity of such an ellipse is nearly unity, and the eccentricity of a parabola is exactly unity.

A comet's path about the sun is, in nearly every respect, similar to that of a projectile relative to the earth. The paths of all known periodic comets are narrow ellipses with the sun in one focus. These bodies are only visible to us when they are near the sun, that is, when they are at one end of their orbits. Consequently the visible portion of these orbits are hardly distinguishable from parabolas. One point in which a projectile's path differs from a comet's is, of course, in size. Another point of difference is that the projectile does not complete its orbit in consequence of the earth's bulk being in its path. The visible portion of a comet's orbit is that near the attracting focus, while the visible portion of a projectile's orbit is that near the "empty" focus.

396. Slow Variation of Circular Motion of a Satellite due to a Resistance.—We shall deal only with the case of a satellite describing a nearly circular orbit. Our object is to find how this orbit will vary if the body moves through a fluid which exerts a small resistance to motion. If a fluid ether fills all space it is natural to expect that this fluid will behave like the other fluids of which we have any knowledge and offer some resistance to the motion of bodies through it.

A small resistance will produce a slow variation of the radius and a corresponding variation of the angular velocity. The effect produced

in one revolution will be assumed to be small, so that many revolutions are performed before the variations are measurable. The rate of variation in the radius is supposed to be so small that the orbit may at any instant be regarded as a circle.

Let F denote the resistance to motion per unit mass of the satellite. We shall suppose F to be constant during the interval we are considering, because only very small variations of distance and velocity are supposed to take place in that interval. Also since the path is nearly circular at every instant we shall assume that the resistance, being opposite to the velocity, is perpendicular to the radius.

If a and n are the radius of the orbit and the angular velocity at the beginning of the interval, r and ω the values of these quantities after a lapse of time t, then the equation of motion along the radius gives

$$\frac{\kappa M}{r^2} = r\omega^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$\frac{\kappa M}{a^2} = an^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence

$$r^3\omega^2 = a^3n^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Also for motion perpendicular to the radius

$$\frac{d}{dt}(r^2\omega) = -r\mathrm{F} = -a\mathrm{F} \text{ nearly} \quad . \quad . \quad . \quad . \quad (4)$$

Therefore

$$r^2\omega = -a\mathrm{F}t + a^2n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Now from (3)

$$r^2\omega^{\frac{4}{3}} = a^2n^{\frac{4}{3}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

From (5) and (6) by division

$$\omega^{-\frac{1}{3}} = -\frac{\mathrm{F}t}{an^{\frac{4}{3}}} + n^{-\frac{1}{3}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Similarly by taking square roots of both sides of (3) and then dividing (5) by the results

$$r^{\frac{1}{2}} = -\frac{\mathrm{F}t}{na^{\frac{1}{2}}} + a^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If we now put $(a + x)$ for r, and $(n + y)$ for ω in (8) and (7), and then retain only the first powers of x and y, these equations give

$$n^{-\frac{1}{3}} - \tfrac{1}{3}n^{-\frac{4}{3}}y = -\frac{\mathrm{F}t}{an^{\frac{4}{3}}} + n^{-\frac{1}{3}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$a^{\frac{1}{2}} + \tfrac{1}{2}a^{-\frac{1}{2}}x = -\frac{\mathrm{F}t}{na^{\frac{1}{2}}} + a^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Hence

$$y = \frac{3\mathrm{F}t}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$x = -\frac{2\mathrm{F}t}{n} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

These show that the angular velocity increases, and the radius decreases in consequence of the resistance. The satellite describes a sort of spiral in which the radius decreases by approximately the same amount in each revolution. Thus the effect of a resistance is to cause the satellite to fall gradually towards the primary.

The linear velocity of the satellite is approximately

$$v = r\omega = \left(a - \frac{2Ft}{n}\right)\left(n + \frac{3Ft}{a}\right)$$

$$= an + Ft \text{ nearly} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Thus the linear velocity of the satellite increases in consequence of the resistance. This is a very remarkable result. The resistance is certainly destroying momentum, but since the satellite falls inwards the component of the attraction along the tangent to the path creates more momentum than the resistance destroys. That, at least, is what our results tell us.

It follows from the above results that the ultimate effect of anything of the nature of fluid resistance to the motion of bodies through space would be to destroy all orbital motion. All satellites would fall into their primaries, and then heat would be generated by the friction which would destroy the relative motion.

397. Variation of Circular Orbit of a Satellite due to Alteration of Mass of Primary.—If M is the mass of the primary, the equation of motion along the radius is

$$\frac{\kappa M}{r^2} = r\omega^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

That is

$$r^3\omega^2 = \kappa M \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Also, since the force acts through a fixed point,

$$r^2\omega = h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

or

$$r^3\omega^{\frac{3}{2}} = h^{\frac{3}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

From (2) and (4)

$$\omega^{\frac{1}{2}} = \frac{k}{h^{\frac{3}{2}}}M \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Taking logarithms of both sides of (5), and then differentiating, we get

$$\frac{1}{2\omega}\cdot\frac{d\omega}{dt} = \frac{1}{M}\cdot\frac{dM}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This gives the relation between the rate of increase of ω and that of M. We can get the rate of alteration of r from (3). Thus on taking logarithms of (3), and then differentiating

$$\frac{2}{r}\cdot\frac{dr}{dt} + \frac{1}{\omega}\cdot\frac{d\omega}{dt} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Whence

$$\frac{1}{r}\cdot\frac{dr}{dt} = -\frac{1}{2\omega}\cdot\frac{d\omega}{dt} = -\frac{1}{M}\cdot\frac{dM}{dt} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Observations seem to indicate that the moon is being accelerated in its orbit, and the acceleration that has not been accounted for is about 12″ per century per century. That is

$$\frac{d\omega}{dt} = 12'' \text{ per century per century} \quad . \quad . \quad (9)$$

This acceleration could be produced by an increase of the earth's mass. If the increase required is small, it might possibly be due to the deposit of meteor showers. We will calculate the required rate of increase of the mass.

The unit of time being a century, the angular velocity of the moon in its orbit is

$$\omega = \frac{360 \times 3600 \times 100 \times 365}{27\cdot3} \text{ seconds per century}$$

$$= 1\cdot73 \times 10^9 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Therefore $\frac{1}{M}\cdot\frac{dM}{dt} = \frac{1}{2\omega}\cdot\frac{d\omega}{dt}$

$$= \frac{12}{2 \times 1\cdot73 \times 10^9} = 3\cdot5 \times 10^{-9} \quad . \quad . \quad . \quad . \quad (10)$$

Taking the average density of the earth to be 350 lbs. per cubic foot, this gives

$$\frac{dM}{dt} = 3\cdot5 \times 10^{-9} \times \tfrac{4}{3}\pi R^3 \times 350 \quad . \quad . \quad . \quad . \quad (11)$$

where R is the radius of the earth.

The deposit on each square foot of the earth's surface per century would have to be

$$\frac{1}{4\pi R^2}\cdot\frac{dM}{dt} = 3\cdot5 \times 10^{-9} \times \tfrac{1}{3}R \times 350 \text{ lbs.}$$

$$= 3\cdot5 \times 10^{-9} \times 4000 \times 1760 \times 350 \text{ lbs.}$$

$$= 8\cdot6 \text{ lbs.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Thus a deposit of 8·6 lbs. per century, or 1·4 ounces per annum on each square foot of the earth's surface would account for the acceleration. This is not a very large amount, and it is not inconceivable that the earth picks up this amount of meteor dust in space.

398. The Problem of Three or more Bodies.—The general problem of three or more bodies has never yet been solved. It is an easy matter to write down equations of motion, and even deduce therefrom a few relations among the co-ordinates and velocities of the bodies, but mathematical analysis is at present unable to furnish the complete solution. It can be proved, as in Art. 388 that, whatever be the forces between the particles of a system of bodies, provided only that the mutual action between each pair is in the line joining them, the centre of mass of the system has no acceleration if no bodies outside the system exert forces on them. It can also be proved that the moment of momentum about any fixed line in space remains constant. It follows

from these principles that, if the influence of the fixed stars be neglected, the centre of mass of the solar system has a constant velocity in space, and that the moment of momentum of the system about any line through this centre of mass remains constant.

But although the general problem cannot be solved, yet under special circumstances an approximate solution can be found. These special circumstances exist everywhere in the solar system. There is no absolutely intractable problem in the motion of any body in the system, although it must be admitted that it is very laborious work to find, with any precision, the motion of a satellite such as the moon.

399. The Moon's Motion.—The force exerted by the sun on a body at the earth's surface is extremely small in comparison with the force exerted by the earth. But at the moon's distance the earth's attraction is very much smaller than at the earth's surface. In fact, the sun's force on the moon is even greater than the earth's force. For, taking the moon's distance from the earth as 240,000 miles, and the earth's distance from the sun as 93,000,000 miles, and denoting the attraction of the sun on unit mass of the earth by F_1, and the attraction of the earth on unit mass of the moon by F_2, we find

$$\frac{F_1}{F_2} = \frac{\text{earth's acceleration towards sun}}{\text{moon's acceleration towards earth}}$$

$$= \frac{9300 \times \Omega^2}{24 \times \omega^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where Ω and ω are the earth's and moon's angular velocities in their orbits.

But $$\frac{\Omega}{\omega} = \frac{27 \cdot 3}{365 \cdot 25} = 0 \cdot 075 \text{ nearly} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Therefore $$\frac{F_1}{F_2} = \frac{9300 \times (0 \cdot 075)^2}{24} = 2 \cdot 17 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Now the moon's distance from the sun is very little different from the earth's distance from the sun. Consequently equation (3) tells us that the sun's force on the moon is about twice the earth's force on the moon.

The earth describes a nearly circular orbit about the sun, and the moon describes a nearly circular orbit about the moving earth, the radius of the earth's orbit being about 389 times the radius of the moon's. Both bodies revolve in the same direction round their primaries. When the moon is at the point of its path nearest to the sun, the resultant force on it is the difference of the sun's and earth's attractions, which is about 1·17 times the earth's attraction, and acts towards the sun. Since this force is perpendicular to the direction of the moon's motion at that point, it is clear that the true path of the moon is concave towards the sun. The moon's path relative to the sun is a curve, winding in and out of the earth's orbit a little over twelve times a-year. But the deviation of the moon's path from the earth's is so small that, when they are represented on the same scale and looked

at from a point where the whole is visible, they can hardly be distinguished. There are no loops on the moon's path; that is, the moon is always travelling in the same direction round the sun as the earth travels. The velocity of the moon relative to the earth is about 0·64 mile per second, and the velocity of the earth relative to the sun is 18·5 miles per second. Thus, even when the moon is at its nearest point to the sun, it is moving in the same direction as the earth with a velocity 18·5 − 0·6, that is, nearly 18 miles per second. The maximum velocity of the moon relative to the sun is 18·5 + 0·6, just over 19 miles per second.

It is clear from the preceding that the moon is just as much a satellite to the sun as to the earth. The sun's controlling force is greater than the earth's. If the earth's force suddenly ceased the moon would go on describing an orbit round the sun pretty much the same as the earth's present orbit; and if the sun's force ceased to act, earth and moon would go off together into space, their relative orbit being very little altered.

400. *A small body coming towards the sun with parabolic velocity passes inside the orbit of a planet just in front of the planet itself, and is deflected by the planet through a right angle, so that when the body has left the planet's influence, it is travelling in nearly the same line as the planet and in the opposite direction. To find the subsequent orbit about the sun.*

The small body has described a nearly hyperbolic orbit about the planet, and it leaves the planet's influence with the same relative velocity as it had on reaching it. If V is the velocity of the planet, the velocity of a body at the same distance from the sun which would describe a parabolic orbit is $\sqrt{2}V$. Since the sphere of influence of the planet is a small region compared with the distance to the sun, we may take $\sqrt{2}V$ to be the velocity of the body when it arrived within this sphere. But since it was supposed to be travelling perpendicular to the direction of motion of the planet, the relative velocity of the body is the vector sum of $\sqrt{2}V$ and V at right angles to each other, that is $\sqrt{3}V$.

When the body passes out of the influence of the planet it leaves with a relative velocity $\sqrt{3}V$. Since it is travelling in the opposite direction to the planet, its velocity relative to the sun is therefore $(\sqrt{3}-1)V$. We have now to find what kind of orbit this body describes about the sun.

If r is the radius of the planet's orbit, and $\frac{k}{r^2}$ the sun's attraction on unit mass, we have, from the assumed circular motion of the planet,

$$V^2 = \frac{k}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If $2a$ denotes the major axis of the new orbit of the small body about the sun

$$\tfrac{1}{2}(\sqrt{3}-1)^2V^2 = \frac{k}{r} - \frac{k}{2a}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

by Art. 357.

Equations (1) and (2) give

$$\frac{1}{2a} = \frac{1}{r} - \frac{(\sqrt{3} - 1)^2}{2r}$$

$$= \frac{1}{2r} 2(\sqrt{3} - 1) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Therefore $$a = \frac{r}{2(\sqrt{3} - 1)} = r\frac{\sqrt{3} + 1}{4} = 0{\cdot}68r \quad . \quad . \quad . \quad . \quad (4)$$

Since the square of the period of revolution is proportional to the cube of the major axis, we find from equation (4) that

$$\frac{\text{Period of small body}}{\text{Period of planet}} = (0{\cdot}68)^{\frac{3}{2}} = 0{\cdot}56 \quad . \quad . \quad . \quad (5)$$

If the disturbing planet is Jupiter, whose period is 11·86 years, the period of the small body would be 6·65 years. It is very probable that Jupiter, with its large mass, has disturbed the orbits of many comets in some such way, and this would explain why so many comets have periods of about 5 or 6 years. If a comet passed into Jupiter's orbit just behind the planet instead of in front of it, and if its direction of motion were turned through a right angle as before, it would escape from Jupiter with a velocity $(\sqrt{3} + 1)V$ relative to the sun, and this velocity, being greater than $\sqrt{2}V$, would carry the comet out of the solar system.

EXAMPLES ON CHAPTER XVIII

1. Assume that the moon's orbit relative to the earth is a circle with radius 240,000 miles, and the earth's orbit about the sun a circle with radius 92 million miles; and assume that the moon makes 13 revolutions in a year in the same direction as the earth's revolutions about the sun. If the earth's attraction on the moon were to cease when the moon is at its greatest distance from the sun, find the ratio of the major axis of the moon's subsequent orbit about the sun to the diameter of the earth's orbit, and find the approximate period in days.

[Ratio = 1·073; period = 406 days.]

2. If the masses of the two components of a binary star are equal, and if their period is 20 years and their distance apart 30 times the distance of the earth from the sun, find the ratio of the mass of each component to the mass of the sun and earth.

[$\frac{270}{8}$.]

3. The star Sirius is one component of a double star, the other component being very faint. Their distance apart is about 28 times the earth's distanee from the sun, and they describe their orbtts about their common centre of mass in 49·4 years. Find the ratio of the sum of their masses to the sum of the masses of the sun and earth.

[Ratio = 9·0 about.]

4. If three equal masses, each equal to the mass of the sun, revolve in the same orbit, each pair being separated by a distance equal to the radius of the earth's orbit, find their common period of revolution.

$$\left[\frac{1}{\sqrt{3}} \text{ of a year.}\right]$$

5. If four masses, each equal to the mass of the sun, describe the same orbit of radius r, the distance between successive masses being $\sqrt{2}r$, where r denotes the radius of the earth's orbit, find the common period of revolution.

$$\left[\frac{2}{\sqrt{2\sqrt{2}+1}} \text{ of a year.}\right]$$

6. Compare the period of revolution of a pair of equal masses about their common centre of mass, under no forces but their mutual attraction, with the period they would have if another equal mass were fixed at their centre of mass.

[The period in the first case is $\sqrt{5}$ times the period in the second.]

7. If r, a, are the radii of the moon's and earth's orbits, show that the maximum relative acceleration of the two bodies due to the sun's attraction on them is approximately $\frac{2r}{a}$ of the earth's acceleration towards the sun.

Assuming thirteen months to a year, find the ratio of this relative acceleration to the acceleration of the moon produced by the earth.

$$\left[\frac{2}{169}.\right]$$

8. Let V denote the velocity of the earth in its orbit, assumed circular, and v the least velocity relative to the earth required to carry a body from its surface to infinity. Find the velocities, relative to the earth, with which a body must be projected from the earth's surface

(1) in the direction of the earth's motion,
(2) in the direction opposite to the earth's motion,
(3) perpendicular to the earth's motion,

so that the body may escape from both the earth and the sun.

[Assume that the relative motion produced by the sun is negligible while the body is so near the earth that the effect of the earth is appreciable; that is, assume that the earth's attraction decreases the kinetic energy of the body relative to the earth, before its distance from the sun has appreciably altered, by as much as if the body went to infinity.]

$$\left[(1)\ \sqrt{v^2+(3-2\sqrt{2})V^2};\ (2)\ \sqrt{v^2+(3+2\sqrt{2})V^2};\ (3)\ \sqrt{v^2+V^2}.\right]$$

9. If a body were projected from the North Pole from a gun pointed at the sun, with the velocity $\sqrt{v^2+V^2}$, v and V having the meanings indicated in the last question, prove that the orbit relative to the sun would be a parabola with the point of projection as one end of the latus rectum, and that its perihelion distance would be half the radius of the earth's orbit.

10. A small body, attracted into the solar system, is approaching the sun with parabolic velocity along a straight line. Its path takes it near Jupiter, and when it gets near enough for the planet's attraction to be appreciable the line of its velocity relative to Jupiter passes at a distance p in front of the planet, this distance being such that the direction of motion relative to Jupiter is turned through a right angle. Find the ratio of p to Jupiter's distance from the sun. Find also the velocity relative to the sun with which

the body escapes from the planet, and calculate the major axis of the new orbit relative to the sun.

[If r = Jupiter's distance from the sun, J his mass, V his velocity, S the mass of the sun, then

$$\frac{p}{r} = \frac{J}{3S}; \quad v^2 = 2(2 - \sqrt{2})V^2; \quad 2a = (\sqrt{2} + 1)r;$$

where v is the velocity of escape, $2a$ the major axis of the new orbit.]

11. If the body mentioned in the last question had its velocity relative to the sun turned through a right angle, through what angle would it be turned relative to Jupiter? Calculate also the velocity of escape from Jupiter, the major axis of the new orbit about the sun, and the period, assuming Jupiter's period to be 11·86 years.

$\left[\text{Angle} = \tan^{-1}(\sqrt{2});\ v = (\sqrt{3} - 1)V;\ 2a = \frac{\sqrt{3}+1}{2}r;\ \text{period} = 6{\cdot}7 \text{ years.}\right]$

12. A small body, attracted into the solar system from rest at a very great distance, passes very near Jupiter and in front of it. At its nearest point its distance from the surface of the planet is equal to three times the radius of the planet. If the diameter of Jupiter subtends 37·4″ at the sun's distance, and if his mass is $\frac{1}{1050}$ of the sun's mass, find the angle through which the direction of motion relative to Jupiter is turned. Also find the angle through which the direction of motion relative to the sun is turned.

[55° 38′ relative to Jupiter, and 92° 8′ relative to the sun.]

13. If the body referred to in the last question passed at a distance from Jupiter equal to the radius of the planet, what would then be the angle through which the direction of motion relative to the sun would be turned? Also calculate the period in the new orbit.

[Angle [illegible]0° 56′; period [illegible] 9·53 years.]

14. A small body which, but for the earth's attraction, would fall into the sun with parabolic velocity, passes in front of the earth and at a distance from the earth's surface equal to the radius of the earth. Find the angle through which the direction of motion relative to the earth is turned, assuming that the radius of the earth and the radius of its orbit are 3960 and 92,000,000 miles respectively, and that the sun's mass is 321,000 times the earth's mass.

[About 1° 22′.]

15. Use the data in question 1 to find what velocity the moon would have relative to the earth if the moon were brought to rest relative to the sun when at its maximum distance from the sun. Use this to find the new orbit the moon would begin to describe relative to the earth.

[About 578 million miles per year; the new orbit would be an hyperbola with eccentricity 868 and major axis $\frac{2}{867}$ of the radius of the moon's orbit. This hyperbola is very nearly a straight line, and it follows that the moon would fall towards the sun scarcely disturbed by the earth's attraction.]

16. Suppose two masses, each equal to the mass of the sun, are at a very great distance apart and travelling in opposite directions with the same speed, v, along a pair of parallel lines which lie at a distance p apart. If V is the velocity of a small body describing a circular orbit of radius p about the sun, show that the bodies pass within the distance

of each other, and that, when they are again at a very great distance apart they are travelling along another pair of parallel lines at distance p apart, but inclined at $2 \cot^{-1} \frac{2v^2}{V^2}$ to the original pair.

17. Suppose one of the bodies in the last question were originally at rest and the other travelling with velocity $2v$ along the line which passes at distance p from the first body. Show that, when the bodies are at a great distance apart after they have passed each other, the body which was at rest moves with velocity

$$\frac{2vV^2}{\sqrt{(4v^4 + V^4)}}$$

parallel to the line joining the foci of the hyperbola it describes relative to the common centre of mass. Prove also that the other body moves along a perpendicular line.

PART III

DYNAMICS OF A RIGID BODY

CHAPTER XIX

EQUATIONS OF MOTION

401. Conception of a Rigid Body.—Just as when finding centres of gravity and moments of inertia, we shall assume that a finite body can be divided up into infinitely small particles touching each other. That is, we take no account of molecular or atomic structure, for we assume that every closed surface, however small, taken inside the mass of a finite body, will enclose some of the mass. In fact, there are supposed to be no empty spaces such as must exist on the molecular hypothesis.

Moreover, in this chapter we are dealing with rigid bodies, and a rigid body is one in which the distance between every pair of particles remains constant. This keeps the size and shape of the body fixed. No body is absolutely rigid, but in dealing with the motion of a body as a whole, the slight alterations in shape of an apparently rigid body are so small that they produce no measurable effect. Consequently most bodies except strings and fluids may be treated as rigid bodies when we wish to find the motion due to given forces.

Except in one or two examples at the end, we shall deal only with the motion of a rigid body parallel to one plane, the xy-plane. The whole body does not necessarily lie in this plane, but every particle moves parallel to it, so that the x and y of any particle vary, but not the z.

402. Motion of the Centre of Mass of a System of Particles whether connected or not.—Let m_1 denote the mass of one of the particles, x_1, y_1, its co-ordinates at any instant, X_1, Y_1, the forces parallel to the axes. Then the equations of motion are

$$m_1\frac{d^2x_1}{dt^2} = X_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$m_1\frac{d^2y_1}{dt^2} = Y_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Now two similar equations can be written down for every particle

of the system. Denoting by the symbol Σ a summation extending to every particle of the system, we get on adding all such equations as (1)

$$\Sigma m\frac{d^2x}{dt^2} = \Sigma X = X, \text{ say} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Similarly, $$\Sigma m\frac{d^2y}{dt^2} = \Sigma Y = Y \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

But, by Art. 86, if $\bar{x}, \bar{y}$, are the co-ordinates of the centre of mass,

$$\Sigma mx = \bar{x}\Sigma m = M\bar{x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where M denotes the total mass.

Differentiating both sides of (5) with respect to t twice in succession

$$\Sigma m\frac{d^2x}{dt^2} = M\frac{d^2\bar{x}}{dt^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Hence equations (3) and (4) may be written

$$M\frac{d^2\bar{x}}{dt^2} = X \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$M\frac{d^2\bar{y}}{dt^2} = Y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

In these equations X and Y denote the sum of all the component forces on all the particles parallel to the axes of x and y respectively. These will generally include the reactions between the particles, and if the particles form a rigid body it would be difficult to find a case in which there would not be reactions between the particles. But these reactions do not appear in X and Y. For suppose m_1 exerts a force R on m_2; then m_2 exerts an equal but opposite force, which we may call $-R$, on m_1. Now the components of R occur in the equations of motion of m_2, and the components of $-R$ occur in the equations of motion of m_1. When these equations are added together the components of R and $-R$ annul each other. In the same way every other reaction between any pair of particles disappears from the sum. Thus X and Y are merely the components of the external forces on the system of particles, that is, the components of the forces exerted by outside bodies on the system.

Equations (7) and (8) are just the same as the equations of motion of a particle of mass M situated at the centre of mass of the system. To find the motion of the centre of mass of any system of particles, whether they form a rigid body or not, we may therefore suppose that the whole mass is concentrated into a particle at the centre of mass, and that all the forces are applied there. For example, the centre of mass of a rigid body thrown up near the earth's surface describes a parabola, and this motion is not affected by any rotation that the body may have. Moreover, if several bodies are thrown into the air at

different times and in different directions, while they are all in the air their common centre of mass describes a parabola.

403. Motion relative to the Centre of Mass. — Multiplying equations (1) and (2) of the last article by $-y_1$ and x_1 respectively, and then adding,

$$m_1\left(x_1\frac{d^2y_1}{dt^2} - y_1\frac{d^2x_1}{dt^2}\right) = x_1Y_1 - y_1X_1 = N_1 \qquad . \quad (1)$$

where N_1 denotes the moment, about the origin, of all the forces acting on the particle m_1.

Now let

$$\left.\begin{aligned} x_1 &= \bar{x} + x_1' \\ y_1 &= \bar{y} + y_1' \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

Then x_1', y_1', are the co-ordinates of m_1 referred to axes parallel to the fixed axes, but always passing through the centre of mass.

Multiplying both sides of the first of equations (2) by m_1, and then summing for all the particles, we get

$$\begin{aligned} \Sigma mx &= \Sigma m\bar{x} + \Sigma mx' \\ &= M\bar{x} + \Sigma mx' \quad . \; . \; . \; . \; . \; . \; . \quad (3) \end{aligned}$$

But we know that

$$\Sigma mx = M\bar{x} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (4)$$

wherever the origin may be. Hence

$$\Sigma mx' = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (5)$$

[1] Differentiating this equation twice with respect to t, we get

$$\Sigma m\frac{d^2x'}{dt^2} = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (6)$$

Now,

$$\Sigma mx\frac{d^2y}{dt^2} = \Sigma\left\{m(\bar{x} + x')\left(\frac{d^2\bar{y}}{dt^2} + \frac{d^2y'}{dt^2}\right)\right\} \quad . \; . \; . \quad (7)$$

The origin for x and y is at any *fixed* point we like to choose. We shall therefore choose this origin at the instantaneous position of the centre of mass, so that, at the instant we are considering,

$$\bar{x} = 0, \bar{y} = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (8)$$

But although $\bar{x}$ and $\bar{y}$ are zero, it does not follow that $\frac{d^2\bar{x}}{dt^2}$ and $\frac{d^2\bar{y}}{dt^2}$ are also zero. These quantities are the components of the acceleration of the centre of mass, and are the same whatever fixed point be taken

[1] If the student has any difficulty in following the process of differentiating a sum indicated by Σ, he is recommended to write down a few terms of the sum and then perform the operation. Thus (5) means

$$m_1x_1' + m_2x_2' + m_3x_3' + \ldots = 0$$

Since this is true for all values of t, we get on differentiating twice

$$m_1\frac{d^2x_1'}{dt^2} + m_2\frac{d^2x_2'}{dt^2} + m_3\frac{d^2x_3'}{dt^2} + \ldots = 0$$

which is the same as equation (6).

as origin. Equations (8) are only true at a particular instant, whereas equation (5), which we differentiated to get (6), is true at all times.

Putting $\bar{x} = 0$ in (7), this gives

$$\begin{aligned}\Sigma mx\frac{d^2y}{dt^2} &= \Sigma mx'\frac{d^2\bar{y}}{dt^2} + \Sigma mx'\frac{d^2y'}{dt^2}\\ &= \frac{d^2\bar{y}}{dt^2}\Sigma mx' + \Sigma mx'\frac{d^2y'}{dt^2}\\ &= 0 + \Sigma mx'\frac{d^2y'}{dt^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)\end{aligned}$$

The zero term in (9) is due to the relation given in (5).

Now summing all such equations as (1) when the fixed origin is taken at the instantaneous position of the centre of mass, we find, on using (9),

$$\Sigma\left\{m\left(x'\frac{d^2y'}{dt^2} - y'\frac{d^2x'}{dt^2}\right)\right\} = \Sigma N = N \quad . \quad . \quad . \quad . \quad (10)$$

where N is used to denote the sum of the moments of the forces on all the particles about the centre of mass of the system.

If r_1, θ_1, are the polar co-ordinates of m_1 referred to the moving centre of mass as pole, it may be shown, as in Art. 348, that

$$x_1'\frac{d^2y_1'}{dt^2} - y_1'\frac{d^2x_1'}{dt^2} = \frac{d}{dt}\left(r_1^2\frac{d\theta_1}{dt}\right) \quad . \quad . \quad . \quad . \quad (11)$$

Thus (10) and (11) give

$$\Sigma m\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = N \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

or

$$\Sigma\frac{d}{dt}\left(mr^2\frac{d\theta}{dt}\right) = N \quad . \quad . \quad . \quad . \quad . \quad (13)$$

A similar result for a single particle has already been proved (equation (3), Art. 348). But equation (13) could not be derived merely by summing for all particles such equations as (3), Art. 348, for in this equation the origin was a fixed point, whereas in (13) the origin is a moving point, namely, the centre of mass of the system.

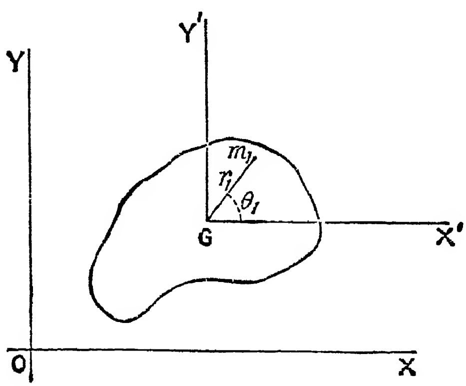


FIG. 174.

Although the reactions between the particles will appear in the equations of the separate particles, yet these reactions will disappear from the sum, as can be seen by the reasoning of the last article. N is

the sum of the moments, about the centre of mass, of the forces acting on all the particles, and since the reactions disappear from the sum, N is therefore the moment of the external forces merely.

The result expressed by (13) may be stated thus—

The rate of increase of the moment of momentum of any system of bodies about an axis through their centre of mass, regarded as a fixed point, is equal to the moment about that axis of all the forces acting on the system.

404. The Moment of Momentum, about an Axis through the Centre of Mass, of a Rigid Body moving parallel to One Plane.—When a rigid body moves parallel to one plane, the motion can be analysed into a motion of the centre of mass and a rotation of the whole body about an axis through the centre of mass and perpendicular to the plane of motion. Any particle m may be considered rigidly attached to this axis by a line perpendicular to the axis, the length of which line is r. Now on account of rigidity all these lines rotate about the axis with the same angular velocity. Also the distance r of any particle from the axis is a constant quantity which does not vary with the time. Thus the moment of momentum of the motion relative to the centre of mass, when the angular velocity is ω, is

$$\Sigma mr^2\omega = \omega\Sigma mr^2 = \mathrm{I}\omega \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where I is the moment of inertia of the body about the axis through the centre of mass.

For a single rigid body the equation (13) of the last article now becomes

$$\mathrm{I}\frac{d\omega}{dt} = \mathrm{N} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The student should satisfy himself that it makes no difference whether we put the differentiation symbol before or after the summation symbol. Thus

$$\sum\frac{d}{dt}\left(mr^2\frac{d\theta}{dt}\right) = \frac{d}{dt}\sum\left(mr^2\frac{d\theta}{dt}\right) \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This, of course, is merely a statement of the fact that the differential coefficient of a sum is the sum of the differential coefficients of the separate terms.

405. Moment of Momentum of a Rigid Body about any Line in Space perpendicular to the Plane of Motion.—Let x, y, be the co-ordinates of any particle m referred to axes through a point on the line fixed in space, and x', y', its co-ordinates referred to parallel axes always passing through centre of mass. Then

$$x = \bar{x} + x' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$y = \bar{y} + y' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Also

$$\Sigma mx' = 0, \ \Sigma my' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

And by differentiating these with respect to t,

$$\Sigma m\frac{dx'}{dt} = 0, \ \Sigma m\frac{dy'}{dt} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Now the moment of momentum about the fixed origin is

$$\Sigma\left\{m\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)\right\}$$

But
$$\Sigma mx\frac{dy}{dt}=\Sigma\left\{m(\bar{x}+x')\left(\frac{d\bar{y}}{dt}+\frac{dy'}{dt}\right)\right\}$$
$$=\Sigma m\bar{x}\frac{d\bar{y}}{dt}+\Sigma mx'\frac{dy'}{dt}+\Sigma mx'\frac{d\bar{y}}{dt}+\Sigma m\bar{x}\frac{dy'}{dt} \quad . \quad . \quad (5)$$

The last two terms in this equation vanish. For, since $\frac{d\bar{y}}{dt}$ is the same quantity in every term of the sum represented by the first of these terms, it follows that

$$\Sigma mx'\frac{d\bar{y}}{dt}=\frac{d\bar{y}}{dt}\Sigma mx' = 0 \text{ by } (3) \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Similarly, the second term in the last line of (5) is zero by (4). Consequently the whole moment of momentum about the fixed origin is

$$\Sigma\left\{m\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)\right\}=\Sigma\left\{m\left(\bar{x}\frac{d\bar{y}}{dt}-\bar{y}\frac{d\bar{x}}{dt}\right)\right\}+\Sigma\left\{m\left(x'\frac{dy'}{dt}-y'\frac{dx'}{dt}\right)\right\} \quad (7)$$

The second sum on the right-hand side of (7) is the moment of momentum about the centre of gravity, the quantity we have previously denoted by $I\omega$. And the first sum on the right-hand side is

$$M\left(\bar{x}\frac{d\bar{y}}{dt}-\bar{y}\frac{d\bar{x}}{dt}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where M is the total mass.

Now the expression (8) is exactly the moment of momentum about the fixed origin of a particle of mass M moving with the centre of mass. Thus the whole moment of momentum about any fixed line perpendicular to the plane of motion is equal to the moment of momentum about a parallel line through the centre of mass together with the moment of momentum of a particle of mass M which is supposed to move with the centre of mass.[1]

406. The Fundamental Equations of Motion of a Rigid Body.— It has been pointed out that the motion of a rigid body parallel to a fixed plane can be analysed into a motion of the centre of mass together with a rotation about an axis through the centre of mass and perpendicular to the plane of motion. Three co-ordinates or geometrical quantities will therefore fix the position of the body. The most convenient quantities to use will generally be the cartesian co-ordinates of the centre of mass, and the angle through which the body has turned from some given position.

[1] See Art. 471, where the momentum is represented as a vector through G together with a momentum-couple $I\omega$.

If we now denote the co-ordinates of the centre of mass by x, y, and the angle through which the body has turned by θ, the three necessary equations of motion are

$$M\frac{d^2x}{dt^2} = X \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$M\frac{d^2y}{dt^2} = Y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$I\frac{d^2\theta}{dt^2} = N \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

These three equations, together with the initial conditions, are sufficient to determine the motion completely.

If N is zero the angular velocity is zero or constant. But if N is zero the resultant force on the body passes through the centre of mass. It follows, then, that if the resultant force on a rigid body always passes through the centre of mass, the angular velocity is constant, and to find the motion of the centre of mass we may treat the body as a particle concentrated at that point.

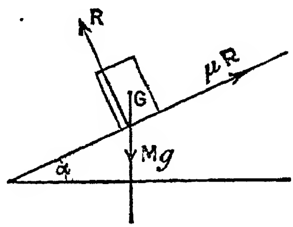


FIG. 175.

407. *A rough circular cylinder slides down an inclined plane with one end in contact with the plane. To determine the friction, the acceleration down the plane, and the condition that this motion should be possible.*

Let α be the inclination of the plane, h the height of the cylinder, r the radius of its base, μ the coefficient of friction, x the displacement of the centre of mass down the plane. Let R be the normal pressure between the cylinder and the plane. Then, for motion down the plane

$$M\frac{d^2x}{dt^2} = Mg \sin \alpha - \mu R \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Since there is no motion perpendicular to the plane,

$$0 = Mg \cos \alpha - R \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and because there is no rotation,

$$0 = \frac{h}{2}\mu R - bR \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where b denotes the perpendicular distance from G, the centre of mass, to the line of action of R, taken positive when G is on the upper side of this line of action.

Now R must act somewhere inside the base, so that b must not be greater than r. Thus the condition that there should be no rotation is, from (3),

$$2r < \text{or} = \mu h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Now multiplying (2) by μ and adding to (1),

$$M\frac{d^2x}{dt^2} = Mg(\sin\alpha - \mu\cos\alpha)$$

or

$$\frac{d^2x}{dt^2} = g(\sin\alpha - \mu\cos\alpha) \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Thus the acceleration down the plane is constant, and equal to $g(\sin\alpha - \mu\cos\alpha)$ when this is positive. If, however, this expression is negative there is no motion, and the assumption made in (1) that the friction is limiting friction is not correct. The expression for the acceleration will be positive when $\tan\alpha$ is greater than μ. This is one of the conditions that the motion we have considered should be possible, and another condition is expressed in (4).

When motion does take place the friction is, from (2),

$$\mu R = \mu Mg\cos\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The displacement and the velocity at any time can be found immediately by integrating (5) when the initial conditions are given.

408. A *and* B *are the points of contact of the front and back wheels of a bicycle with a level road,* G *is the centre of mass of rider and bicycle.* G *is at a height* c *above the line* AB, *and the vertical through* G *divides* AB *into two parts of lengths* a *and* b. *If the rotation of the front wheel is stopped by the brake, the back wheel not being braked, and if the coefficient of friction between the wheel and the ground is* μ, *to find the pressures between the wheels and the ground, and to compare them with the pressures before the brake was applied.*

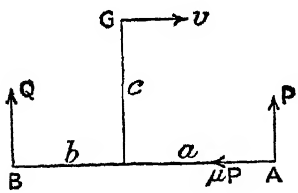


FIG. 176.

Since there is no vertical acceleration, we have

$$0 = Mg - P - Q \quad . \quad (1)$$

M denoting the mass of the rider and bicycle. Also, since there is no rotation, the moment of the forces about the centre of mass is zero; that is,

$$0 = aP - c\mu P - bQ \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Solving (1) and (2) for P and Q, we get

$$P = \frac{b}{a+b-c\mu}Mg \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$Q = \frac{a-c\mu}{a+b-c\mu}Mg \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The pressures before the brake was applied were

$$P_0 = Mg\frac{b}{a+b} = \frac{a+b-c\mu}{a+b}P \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$Q_0 = Mg\frac{a}{a+b} = \frac{a(a+b-c\mu)}{(a+b)(a-c\mu)}Q \quad . \quad . \quad . \quad . \quad (6)$$

If the expression for Q in (4) is negative, the motion we have assumed is not possible. The bicycle will kick in this case, and the rider will be thrown over the front wheel. The condition that this should not happen is

$$Q > 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

that is,

$$a > c\mu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

In an ordinary bicycle the following are rough values of the quantities used—

$$b = 20 \text{ inches}, \; a = 25 \text{ inches}, \; c = 44 \text{ inches}, \; \mu = 0{\cdot}58 \qquad (9)$$

Therefore

$$a - c\mu = 25 - 25{\cdot}5 \text{ inches} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

which is nearly zero. It follows, therefore, that if the brake be forcibly applied to the front wheel there is a great danger of kicking. In this case the whole weight is thrown on the front wheel.

The horizontal force is μP, and this produces a horizontal retardation

$$\frac{\mu P}{M} = \frac{\mu b}{a + b - \mu c} g \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

which, with the numerical values in (5), is nearly μg or $0{\cdot}58g$.

It is left as an exercise for the student to prove that if the back wheel instead of the front wheel is stopped by the brake, the pressures are

$$P = \frac{b + \mu c}{a + b + \mu c} Mg \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

$$Q = \frac{a}{a + b + \mu c} Mg \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Also the horizontal retardation is

$$\frac{\mu Q}{M} = \frac{\mu a}{a + b + \mu c} g = 0{\cdot}206g \text{ nearly} \quad . \quad . \quad . \quad . \quad (14)$$

409. *A string is wrapped round a solid cylinder of radius* a, *and to the free end a mass is attached. The cylinder is free to rotate about its axis, which is horizontal. To find the motion due to the falling of the mass attached to the string.*

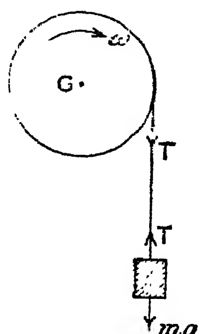


FIG. 177.

Let M and m denote the masses of the cylinder and the body attached to the string.

When the angular velocity of the cylinder is ω, the linear velocity of m is $a\omega$. Denoting the tension in the string by T, the equation of motion of m is

$$m\frac{d(a\omega)}{dt} = mg - T \quad . \quad . \quad . \quad (1)$$

The equation for the rotational motion of the cylinder is

$$I\frac{d\omega}{dt} = aT \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Multiplying (1) by a and adding to (2), we get

$$(I + ma^2)\frac{d\omega}{dt} = amg \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

whence
$$\frac{d\omega}{dt} = \frac{amg}{I + ma^2} = \frac{amg}{\frac{1}{2}Ma^2 + ma^2} = \frac{2mg}{(M + 2m)a} \quad . \quad . \quad (4)$$

Thus the angular acceleration is constant, and therefore the linear acceleration of m is constant. If θ is the angular displacement of the cylinder since it was at rest, and t is reckoned from this instant, then by integrating (4) twice,

$$\theta = \frac{mg}{(M + 2m)a}t^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

If x denotes the distance through which m has dropped in this time

$$x = a\theta = \frac{mg}{M + 2m}t^2$$

410. *A sphere, whose centre of mass is at its centre of figure, rolls down a rough inclined plane without sliding. To find the distance travelled in* t *seconds from rest.*

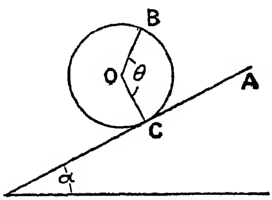


FIG. 178.

Suppose that, at the beginning of the motion, B was in contact with A. C is the point of contact t seconds after the start. Let AC be denoted by x, and let a be the radius. Let R be the normal pressure between the sphere and the plane, F the friction up the plane. Then, for the motion of the centre of mass parallel to the plane,

$$M\frac{d^2x}{dt^2} = Mg \sin \alpha - F \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

And since there is no motion perpendicular to the plane,

$$0 = Mg \cos \alpha - R \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

For the rotational motion

$$Mk^2\frac{d^2\theta}{dt^2} = aF \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

k being the radius of gyration of the sphere.

Since there is no sliding, AC is equal to the arc BC; that is,

$$x = a\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Differentiating both sides twice, we get

$$\frac{d^2x}{dt^2} = a\frac{d^2\theta}{dt^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Eliminating F from (1) and (3),

$$M\left(a\frac{d^2x}{dt^2} + k^2\frac{d^2\theta}{dt^2}\right) = aMg \sin \alpha \quad . \quad . \quad . \quad (6)$$

From (5) and (6)

$$\frac{d^2x}{dt^2} = \frac{a^2}{a^2 + k^2} g \sin \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The distance travelled in t seconds from rest is thus

$$x = \frac{1}{2} \cdot \frac{a^2}{a^2 + k^2} gt^2 \sin \alpha \quad . \quad . \quad . \quad . \quad . \quad (8)$$

When a body slides down a smooth plane its acceleration is $g \sin \alpha$. Thus the acceleration of the rolling body is $\frac{a^2}{a^2 + k^2}$ of the acceleration of a sliding body.

For a solid sphere

$$k^2 = \tfrac{2}{5}a^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and therefore

$$\frac{d^2x}{dt^2} = \tfrac{5}{7}g \sin \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

For a hollow sphere with inner and outer radii b and a,

$$k^2 = \frac{2}{5} \cdot \frac{a^5 - b^5}{a^3 - b^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

The same method will clearly apply to a cylinder.

For a solid cylinder

$$k^2 = \tfrac{1}{2}a^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

and the distance travelled from rest is

$$x = \tfrac{1}{3}gt^2 \sin \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Equation (1) now gives the friction. Thus

$$F = Mg \sin \alpha - M\frac{d^2x}{dt^2}$$

$$= \frac{k^2}{a^2 + k^2} Mg \sin \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

411. *A sphere or cylinder partly rolls and partly slides down a rough inclined plane. To find the acceleration down the plane and the angular acceleration.*

When the body is rolling and sliding the friction is limiting friction.

The equations (1), (2), and (3), of the last article are still true. We will write them here again—

$$M\frac{d^2x}{dt^2} = Mg \sin \alpha - F \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$0 = Mg \cos \alpha - R \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$Mk^2\frac{d^2\theta}{dt^2} = aF \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

But since sliding takes place the equation (4) is no longer true. In its stead we have the equation expressing the fact that the friction is limiting friction. Thus

$$F = \mu R \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

From equation (2),

$$\mu R = \mu M g \cos \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Substituting this value for F in (1) and (3), and then dividing each by M,

$$\frac{d^2x}{dt^2} = g \sin \alpha - \mu g \cos \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$k^2 \frac{d^2\theta}{dt^2} = \mu a g \cos \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Thus both the acceleration of the centre of mass and the angular acceleration are constant. The motion of the centre of mass is entirely independent of the form of the body when sliding does occur. The equation (6) is just the same for a body which does not roll at all.

412. The Test for Sliding when a Cylinder or Sphere rolls down an Inclined Plane.—When a sphere or cylinder rolls down an inclined plane the motion may be pure rolling, or a combination of rolling and sliding, and we have investigated both sorts of motion. But we want some test which will enable us to discover which is the correct kind of motion in any particular case. This test will now be given.

Equations (2) and (14) of Art. 410 give

$$\frac{F}{R} = \frac{k^2}{a^2 + k^2} \tan \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now F cannot be greater than μR. Consequently the motion will be a combination of rolling and sliding if

$$\frac{k^2}{a^2 + k^2} \tan \alpha > \mu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and it will be pure rolling if

$$\frac{k^2}{a^2 + k^2} \tan \alpha < \mu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Before attempting to solve the equations of motion for a body rolling down a plane, it will therefore be necessary to discover which of the inequalities (2) or (3) is true. If (2) is true the results in Art. 411 are correct, but if (3) is true the results in Art. 410 must be used.

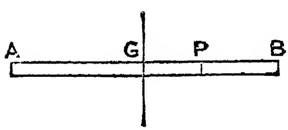


FIG. 179.

413. *A uniform rod of length* 2a *is spinning in a horizontal plane about an axis through its centre of gravity. To find the tension at a distance* 2x *from one end.*

Let m be the mass of unit length of the rod, ω the angular velocity, T the tension required.

Let AB be the rod, P the point at which the tension is required, the distance PB being $2x$.

The centre of mass of the portion PB describes a circle of radius $(a - x)$. Hence this point has an acceleration $(a - x)\omega^2$ towards the centre G of the circle. The mass of PB is $2xm$. Hence, resolving along BG

$$2xm(a - x)\omega^2 = \mathrm{T} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This is the tension required.

We should get exactly the same result, of course, by considering the motion of the portion AP. For its centre of mass describes a circle of radius x, and its mass is $2(a - x)m$. Hence

$$2(a - x)m \,.\, x\omega^2 = \mathrm{T} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which is the same as before.

414. *A circular hoop of radius* a *and mass* m *per unit length rotates with angular velocity* ω *radians per second about its axis, which is vertical. To find the tension in the hoop.*

We will consider the motion of half of the hoop whose centre of mass is at G. Now if C is the centre of the hoop

$$\mathrm{CG} = \frac{2a}{\pi} \quad . \quad . \quad . \quad (1)$$

The acceleration of G is $\mathrm{CG}\,.\,\omega^2$ towards C, and the mass of half of the hoop is πma. Hence, resolving along GC for the motion of this half,

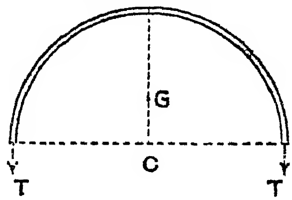


FIG. 180.

$$\pi ma \,.\, \mathrm{CG} \,.\, \omega^2 = 2\mathrm{T} \quad . \quad . \quad (2)$$

since the only horizontal forces on the mass we are considering are the tensions of the other half at the plane of separation of the two halves.

From (2) $\qquad \mathrm{T} = ma^2\omega^2 = mv^2$ poundals . . . . . (3)

where v is the velocity of a point on the hoop.

If s is the area of the section of the hoop, and ρ the density of the material, then, s being the volume of unit length,

$$m = \rho s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Hence $\qquad \dfrac{\mathrm{T}}{s} = \rho v^2 = \dfrac{\rho v^2}{g}$ lbs. . . . . (5)

Now $\dfrac{\mathrm{T}}{s}$ is the tension across unit area of the section of the hoop. For cast iron the breaking tension is about 432×10^4 lbs. per square foot, or $432 \times 32 \times 10^4$ poundals per square foot, and the density is 450 lbs. per cubic foot. If, then, a cast-iron hoop is just at breaking-point

$$432 \times 32 \times 10^4 = 450v^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

from which $\qquad v = 554$ feet per sec. $= 6{\cdot}3$ miles per min. . (7)

If the actual tension of the hoop were one-ninth of the breaking tension, then we should have

$$v = \frac{6{\cdot}3}{3} = 2{\cdot}1 \text{ miles per min.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

415. *A solid homogeneous sphere of radius* a *is projected along a rough horizontal plane with a velocity* v_0. *When it is projected it is spinning about the horizontal diameter perpendicular to* v_0 *with an angular velocity* ω_0 *in the direction contrary to that in which it would spin if there were no sliding. To find the ultimate motion.*

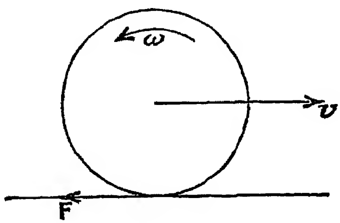


FIG. 181.

Let v and ω be the velocity of the centre of mass and the angular velocity t seconds after the projection. Let F denote the friction acting on the sphere.

Then resolving horizontally,

$$m\frac{dv}{dt} = -\mathrm{F} \quad . \quad . \quad (1)$$

The equation of rotation about the centre of mass is

$$\mathrm{I}\frac{d\omega}{dt} = -a\mathrm{F} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Putting $\frac{2}{5}a^2m$ for I in (2), we get

$$\tfrac{2}{5}ma\frac{d\omega}{dt} = -\mathrm{F} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Now from (1) and (3)

$$\tfrac{2}{5}a\frac{d\omega}{dt} - \frac{dv}{dt} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Integrating this $\qquad \frac{2}{5}a\omega - v = \text{a constant}$

$$= \tfrac{2}{5}a\omega_0 - v_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The sliding will continue until $v = -a\omega$. When this happens either ω or v will have changed its sign, and the motion will be one of pure rolling in one direction or the other.

When the sliding has ceased we find from (5)

$$\tfrac{7}{5}a\omega = -\tfrac{7}{5}v = \tfrac{2}{5}a\omega_0 - v_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This shows that the sphere will come to rest, and then begin to move backwards if

$$\tfrac{2}{5}a\omega_0 - v_0$$

is positive. But if this quantity is negative the spin will stop, and then the sphere will begin to spin in the opposite direction, and finally it will roll without sliding in the same direction as v_0.

416. To find the tension, shearing force, and bending moment at any point of a uniform rod moving in any manner in one plane.

Let AB be the rod, P the point at which the stresses are required. Let PA = z.

Let α and β denote the component accelerations of G, the centre of mass of PA, along and perpendicular to PA. Let X and Y denote the components of the external forces on PA in the directions of α and β; and let N denote the moment of these external forces about G. Let m denote the mass of unit length of the rod. Let F denote the shear, T the tension, and M the bending moment.

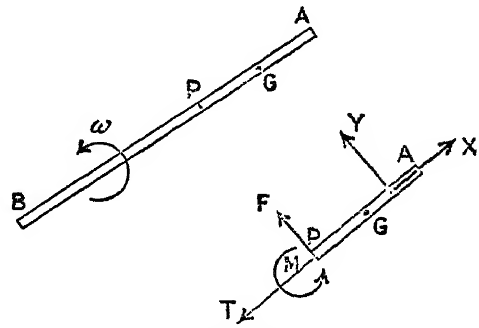


FIG. 182.

For the motion of the centre of mass of PA we have the two equations

$$zm \,.\, \alpha = X - T \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$zm \,.\, \beta = Y + F \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If ω denotes the angular velocity of the rod, the equation for rotation of PA about G is

$$I\frac{d\omega}{dt} = M + N - \tfrac{1}{2}zF \quad . \quad . \quad . \quad . \quad . \quad (3)$$

But $$I = \tfrac{1}{12}z^2 \,.\, mz$$

Hence $$\tfrac{1}{12}mz^3\frac{d\omega}{dt} = M + N - \tfrac{1}{2}zF \quad . \quad . \quad . \quad . \quad . \quad (4)$$

When α, β, and $\dfrac{d\omega}{dt}$, are known, equations (1), (2), and (4), give the stresses T, F, and M, in terms of X, Y, and N, which are supposed to be known forces.

417. *A uniform rod turns without friction about a horizontal axis through one end under no forces except the action at the hinge and its own weight. If its angular velocity in the highest position is ω_0, to find the angular velocity in any other position and the reactions of the axis.*

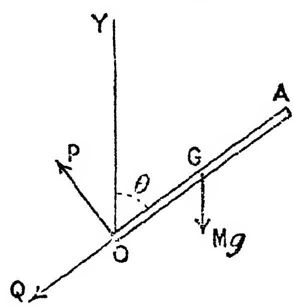


FIG. 183.

Let θ be the angle which the rod makes with the vertical at any instant, P and Q the actions of the hinge perpendicular to, and along, the rod respectively. Let l be the length of the rod. The

acceleration of G perpendicular to the rod, in the opposite direction from that in which P is shown, is

$$\frac{d^2}{dt^2}\left(\frac{l\theta}{2}\right) = \tfrac{1}{2}l\frac{d^2\theta}{dt^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Hence, if M is the mass of the rod,

$$\tfrac{1}{2}\mathrm{M}l\frac{d^2\theta}{dt^2} = \mathrm{M}g\sin\theta - \mathrm{P} . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

And for rotation about the centre of mass

$$\tfrac{1}{12}\mathrm{M}l^2\frac{d^2\theta}{dt^2} = \frac{l}{2}\mathrm{P} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

since the moment of inertia about G is $\frac{1}{12}\mathrm{M}l^2$. Eliminating P from (2) and (3), we get

$$\tfrac{1}{8}\mathrm{M}l^2\frac{d^2\theta}{dt^2} = \frac{l}{2}\mathrm{M}g\sin\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

or

$$2l\frac{d^2\theta}{dt^2} = 3g\sin\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Writing ω for $\frac{d\theta}{dt}$, then

$$\frac{d^2\theta}{dt^2} = \frac{d\omega}{dt} = \frac{d\omega}{d\theta}\cdot\frac{d\theta}{dt} = \omega\frac{d\omega}{d\theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Therefore (5) can be written

$$2l\omega\frac{d\omega}{d\theta} = 3g\sin\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Integrating this with respect to θ

$$l\omega^2 = -3g\cos\theta + c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Since $\omega = \omega_0$ when $\theta = 0$, we get

$$l\omega_0^2 = -3g + c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Subtracting (9) from (8)

$$l\omega^2 - l\omega_0^2 = 3g(1 - \cos\theta) . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Using the value of the angular acceleration in (5), equation (3) gives

$$\mathrm{P} = \tfrac{1}{4}\mathrm{M}g\sin\theta = \tfrac{1}{4}\mathrm{W}\sin\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

where W is the weight of the rod.

To find Q, resolve along GO for the motion of the centre of mass. Thus, since G describes a circle of radius $\frac{1}{2}l$ its acceleration along GO is $\frac{1}{2}l\omega^2$. Hence

$$\tfrac{1}{2}\mathrm{M}l\omega^2 = \mathrm{Q} + \mathrm{M}g\cos\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

or

$$\begin{aligned}\mathrm{Q} &= \tfrac{1}{2}\mathrm{M}l\omega^2 - \mathrm{M}g\cos\theta \\ &= \tfrac{1}{2}\mathrm{M}l\omega_0^2 + \tfrac{3}{2}\mathrm{M}g(1-\cos\theta) - \mathrm{M}g\cos\theta \\ &= \tfrac{1}{2}\mathrm{M}l\omega_0^2 + \tfrac{1}{2}\mathrm{M}g(3 - 5\cos\theta) \\ &= \tfrac{1}{2}\mathrm{W}\left\{\frac{l\omega_0^2}{g} + (3 - 5\cos\theta)\right\} \quad . \quad . \quad . \quad . \quad (13)\end{aligned}$$

The kinetic energy of the whole body is therefore

$$\tfrac{1}{2}\Sigma m(u^2 + v^2) = \tfrac{1}{2}M(\bar{u}^2 + \bar{v}^2) + \tfrac{1}{2}\Sigma m\left\{\left(\frac{dx'}{dt}\right)^2 + \left(\frac{dy'}{dt}\right)^2\right\} \quad . \quad (9)$$

Now the quantity involving Σ on the right hand of (9) is the kinetic energy of the motion relative to the centre of mass, and is the same whether the centre of mass is in motion or not, provided the relative motion remains the same. But the only possible motion for a rigid body relative to the centre of mass is one of rotation. Since the motion is parallel to one plane, the body therefore rotates about an axis through the centre of mass perpendicular to this plane.

Let r be the distance of m from this axis through the centre of mass, ω the angular velocity of the body. Then clearly the velocity of m relative to the centre of mass is $r\omega$. Hence

$$r^2\omega^2 = \left(\frac{dx'}{dt}\right)^2 + \left(\frac{dy'}{dt}\right)^2 \quad . \quad . \quad . \quad . \quad (10)$$

Thus (9) and (10) give

$$\begin{aligned} \tfrac{1}{2}\Sigma m(u^2 + v^2) &= \tfrac{1}{2}M(\bar{u}^2 + \bar{v}^2) + \tfrac{1}{2}\Sigma mr^2\omega^2 \\ &= \tfrac{1}{2}M(\bar{u}^2 + \bar{v}^2) + \tfrac{1}{2}\omega^2\Sigma mr^2 \\ &= \tfrac{1}{2}M(\bar{u}^2 + \bar{v}^2) + \tfrac{1}{2}I\omega^2 \quad . \quad . \quad . \quad . \quad (11) \end{aligned}$$

This, then, is the kinetic energy of the rigid body.

If one point at a distance h from the centre of mass is fixed, then the resultant velocity V of the centre of mass is $h\omega$. Hence

$$\bar{u}^2 + \bar{v}^2 = V^2 = h^2\omega^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Therefore the kinetic energy is

$$\tfrac{1}{2}Mh^2\omega^2 + \tfrac{1}{2}I\omega^2 = \tfrac{1}{2}(Mh^2 + I)\omega^2 \quad . \quad . \quad . \quad . \quad (13)$$

But the quantity $Mh^2 + I$ is the moment of inertia about the fixed axis, and if we denote this by I′, the kinetic energy becomes

$$\tfrac{1}{2}I'\omega^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

This result can be obtained easily without using the result given in (11). For the velocity of a particle m at a distance r' from the fixed axis is $r'\omega$. Hence its kinetic energy is $\tfrac{1}{2}mr'^2\omega^2$. Thus the kinetic energy of the whole body is

$$\Sigma\tfrac{1}{2}mr'^2\omega^2 = \tfrac{1}{2}\omega^2\Sigma mr'^2 = \tfrac{1}{2}\omega^2 I' \quad . \quad . \quad . \quad . \quad (15)$$

However a body is moving parallel to one plane the motion at each instant can be represented by a rotation about the instantaneous axis of rotation. This, at any rate, gives the correct velocity at every point of the rigid body. Consequently, if I′ is the moment of inertia about the instantaneous axis, the kinetic energy is $\tfrac{1}{2}I'\omega^2$ as in (14). When we

know the position of the instantaneous axis, this last form for the kinetic energy is very useful. Moreover, the use of the instantaneous axis gives us an alternative proof of (11).

420. Energy Equation for a Rigid Body.—If the external forces acting on a rigid body be replaced by a force with components X and Y acting at the centre of mass, together with a couple N, then the work done in any displacement of the body is, by Art. 128,

$$\int X dx + \int Y dy + \int N d\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where x and y are the co-ordinates of the centre of mass, and θ is the angle which some plane in the body perpendicular to the plane of motion makes with another plane fixed in space also perpendicular to the plane of motion.

Now the total work done on the rigid body by the external forces is clearly the sum of the works done on its separate particles, because the work done by the reactions between the particles will disappear from the equations when we sum for all the particles, just as the reactions themselves disappeared from the equations of motion. But the increase in the kinetic energy of each particle in any interval is equal to the work done on it in that interval. Consequently the increase in the kinetic energy of the rigid body in any interval is equal to the work done by the external forces. That is, if V is the resultant velocity of the centre of mass

$$\tfrac{1}{2}MV^2 + \tfrac{1}{2}I\omega^2 = \int X dx + \int Y dy + \int N d\theta + C \quad . \quad . \quad (2)$$

The constant C represents the kinetic energy at the beginning of the displacement considered.

If several rigid bodies are attached together so that the reactions between them can do no work, then the sum of their kinetic energies is equal to the work done on them by the external forces in any interval of time, together with their kinetic energies at the beginning of that interval.

421. Energy Equation by another Method.—We can obtain the energy equation (2) of the last article immediately by integrating the equations of motion of the rigid body. These equations are

$$M\frac{du}{dt} = X \quad . \quad . \quad . \quad . \quad (1)$$

$$M\frac{dv}{dt} = Y \quad . \quad . \quad (2)$$

$$I\frac{d\omega}{dt} = N \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where u and v are the component velocities of the centre of mass. Now

$$\frac{du}{dt} = \frac{du}{dx} \cdot \frac{dx}{dt} = u\frac{du}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The other two accelerations can be similarly transformed. Then

$$Mu\frac{du}{dx} = X \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$Mv\frac{dv}{dy} = Y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$I\omega\frac{d\omega}{d\theta} = N \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Integrating these with respect to x, y, and θ, and adding the results

$$\tfrac{1}{2}M(u^2 + v^2) + \tfrac{1}{2}I\omega^2 = \int Xdx + \int Ydy + \int Nd\theta + C \quad . \quad (8)$$

which is the same equation as before.

422. Energy Equation when the External Forces have a Potential.—If the external forces have a potential U, then the work done in a small displacement is $-d$U. Thus

$$-U = \int Xdx + \int Ydy + \int Nd\theta + A \quad . \quad . \quad \quad . \quad (1)$$

Hence, V being the resultant velocity of the centre of mass of the rigid body,

$$\tfrac{1}{2}MV^2 + \tfrac{1}{2}I\omega^2 = -U + \text{a constant} \quad . \quad . \quad . \quad (2)$$

or

$$\tfrac{1}{2}MV^2 + \tfrac{1}{2}I\omega^2 + U = \text{a constant} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

423. Degrees of Freedom.—If n geometrical quantities must be given in order to fix the position of a body or system of bodies, that body or system is said to have n degrees of freedom. Thus a particle moving in space has three degrees of freedom, since three co-ordinates x, y, z, are needed to fix its position. But if the particle is only free to move parallel to one plane, say the xy plane, then one co-ordinate is fixed and only two are left free; that is, only two degrees of freedom remain.

If a rigid body is free to move parallel to a fixed plane it has three degrees of freedom. For its position can be fixed by the two co-ordinates of the centre of mass and the angle θ through which it has rotated. Or instead of θ the third geometrical quantity might be the abscissa of any particle in the body except the centre of mass. Or again the body can be fixed by the abcissæ of any two particles in the body and the ordinate of a third point. In fact, any three independent co-ordinates or geometrical quantities depending on the position of the body will fix that position.

If a rigid body has an axis fixed so that it can only rotate about that axis, then it has only one degree of freedom. Or if a body rolls without sliding, parallel to a given plane, on any fixed surface, it has only one degree of freedom.

424. Sufficiency of the Energy Equation when there is only one Degree of Freedom.—When a body has only one degree of freedom—that is, when only one geometrical quantity is needed to fix its position—the equations of motion may generally be omitted and the energy

equation used in their stead. For all the velocities can be expressed in terms of the rate of increase of one geometrical quantity, and thus the kinetic energy can be expressed in terms of the rate of increase of this quantity. If we can easily find the work done the equation of energy can be written down at once, and this equation will lead to the geometrical quantity which fixes the position of the body.

Even in cases where we have to find some of the forces acting on the body, and it is necessary to get these forces from the equations of motion, it will often be found easiest to get the velocities (and therefore the accelerations by differentiation) from the energy equation.

425. *A uniform rod is placed in a vertical position with one end on a smooth horizontal floor. It is then let go, and it falls to the floor from rest in the vertical position. To find its angular velocity in any position and the pressure on the floor.*

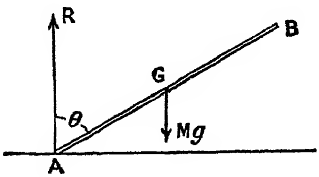


FIG. 184.

The rod would be in equilibrium in the vertical position, and it would only begin to move when it was slightly disturbed from the vertical position. It could not therefore fall exactly from rest in the vertical position. Nevertheless, the disturbance which started the motion may be considered to be so small that it may be neglected in our equations.

If v is the downward velocity of the centre of mass G, and ω the angular velocity, then so long as the end A is in contact with the floor,

$$v = \frac{d}{dt}\left\{\frac{l}{2}(1 - \cos\theta)\right\} = \frac{l}{2}\sin\theta \cdot \omega \quad . \quad . \quad . \quad . \quad (1)$$

where l is the length of the rod.

Now as there are no horizontal forces on the rod, and there was no initial horizontal motion, the centre of mass G moves in a vertical line. If M is the mass of the rod, the moment of inertia about G is $\frac{1}{12}l^2\mathrm{M}$. Consequently the energy equation is

$$\tfrac{1}{2}\mathrm{M}v^2 + \tfrac{1}{24}l^2\mathrm{M}\omega^2 = \tfrac{1}{2}\mathrm{M}gl(1 - \cos\theta) \quad . \quad . \quad . \quad (2)$$

the right-hand side being the work done by the weight Mg as the centre of mass falls from a height $\frac{l}{2}$ to a height $\frac{l}{2}\cos\theta$.

Using (1), equation (2) gives

$$(3\sin^2\theta + 1)l^2\omega^2 = 12gl(1 - \cos\theta) \quad . \quad . \quad (3)$$

Now we can find R by using the equation for rotational motion. Thus

$$\tfrac{1}{12}\mathrm{M}l^2\frac{d\omega}{dt} = \frac{l}{2}\sin\theta \cdot \mathrm{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Now, by a familiar transformation,

$$\frac{d\omega}{dt} = \omega\frac{d\omega}{d\theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

But from (3) $$l^2\omega^2 = 12gl\frac{1-\cos\theta}{1+3\sin^2\theta} = 12gl\frac{1-\cos\theta}{4-3\cos^2\theta} \quad . \quad . \quad (6)$$

Therefore $$2l^2\omega\frac{d\omega}{d\theta} = 12gl\frac{\sin\theta(4-6\cos\theta+3\cos^2\theta)}{(4-3\cos^2\theta)^2} \quad . \quad . \quad (7)$$

Equation (4) now gives

$$R = Mg\frac{4-6\cos\theta+3\cos^2\theta}{(4-3\cos^2\theta)^2}$$

$$= W\frac{4-6\cos\theta+3\cos^2\theta}{(4-3\cos^2\theta)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

W being the weight of the rod.

We have assumed throughout that the end A will always remain in contact with the floor. We can now test whether this assumption is correct or not. If the expression for R in (8) becomes negative before $\theta = \frac{\pi}{2}$, then the end A will rise off the floor because the floor cannot exert a pull, which is what is meant by a negative value of R. But the numerator of the fraction on the right-hand side of (8) cannot be negative for any value of $\cos\theta$. This numerator can be written

$$3(1-\cos\theta)^2 + 1$$

which is clearly always positive.

At the beginning of the motion, when $\theta = 0$, we find $R = W$. In the horizontal position just before B strikes the floor $R = \frac{1}{4}W$.

426. *A sphere or cylinder rolls without sliding in a vertical plane inside a rough fixed horizontal cylinder. To find the velocity at any point and also the normal pressure and the friction.*

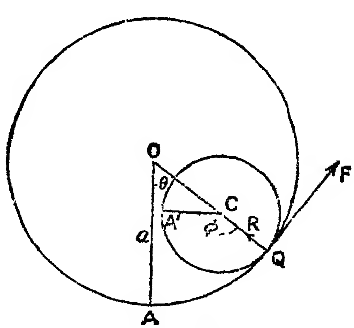


FIG. 185.

Let a be the radius of the fixed cylinder, b the radius of the rolling body. A is the lowest point of the fixed cylinder, A′ the point of the rolling cylinder which was in contact with A.

The angle which the rolling body has turned through from the equilibrium position is the angle of inclination of CA′ to the vertical, because CA′ was vertical in the equilibrium position. This angle is $(\phi - \theta)$. Now, on account of the rolling motion,

$$a\theta = b\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Hence $$\phi - \theta = \frac{a-b}{b}\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The angular velocity of the rolling body is thus

$$\omega = \frac{d}{dt}(\phi - \theta) = \frac{a - b}{b}\frac{d\theta}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The centre of mass C of the rolling body describes a circle of radius $(a - b)$. Hence its linear velocity is

$$v = (a - b)\frac{d\theta}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The only force which does any work on the rolling body is its weight, for the point Q at which the other forces are applied is at rest.

If k is the radius of gyration of the rolling body about the axis through C perpendicular to the plane of motion, the energy equation is

$$\tfrac{1}{2}Mv^2 + \tfrac{1}{2}Mk^2\omega^2 = -Mg(a - b)(1 - \cos\theta) + \text{a constant} \quad (5)$$

On dividing by $\frac{M}{2}$ and substituting for v and ω, this gives

$$(a - b)^2\left(\frac{d\theta}{dt}\right)^2\left(\frac{b^2 + k^2}{b^2}\right) = 2g(a - b)\cos\theta + K \quad . \quad . \quad (6)$$

Now let R and F denote the normal pressure and the friction. Then resolving along the line CO for the motion of centre of mass

$$M(a - b)\left(\frac{d\theta}{dt}\right)^2 = R - Mg\cos\theta \quad . \quad . \quad . \quad . \quad (7)$$

Therefore
$$R = Mg\cos\theta + M(a - b)\left(\frac{d\theta}{dt}\right)^2$$
$$= Mg\cos\theta + 2Mg\frac{b^2}{b^2 + k^2}\cos\theta + \frac{MK}{a - b}\cdot\frac{b^2}{b^2 + k^2}$$
$$= Mg\frac{3b^2 + k^2}{b^2 + k^2}\cos\theta + \frac{MK}{a - b}\cdot\frac{b^2}{b^2 + k^2} \quad . \quad . \quad . \quad (8)$$

We can find F either from the equation for rotational motion or from the equation of motion of the centre of mass perpendicular to CO. We shall use this second equation. Thus

$$M(a - b)\frac{d^2\theta}{dt^2} = F - Mg\sin\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

That is
$$F = M(a - b)\frac{d^2\theta}{dt^2} + Mg\sin\theta \quad . \quad . \quad . \quad (10)$$

On differentiating both sides of (6) with respect to t, we get

$$2(a - b)^2\frac{d\theta}{dt}\cdot\frac{d^2\theta}{dt^2}\cdot\frac{b^2 + k^2}{b^2} = -2g(a - b)\sin\theta\frac{d\theta}{dt} \quad . \quad (11)$$

Hence
$$(a - b)\frac{d^2\theta}{dt^2} = -g\frac{b^2}{b^2 + k^2}\sin\theta \quad . \quad . \quad . \quad (12)$$

Therefore
$$F = Mg\sin\theta\frac{k^2}{b^2 + k^2} \quad . \quad . \quad . \quad (13)$$

The constant K in (6) and (8) depends on the initial conditions. If n is the value of $\frac{d\theta}{dt}$ when θ is a right angle, then (6) shows that

$$K = (a - b)^2 n^2 \frac{b^2 + k^2}{b^2} \quad . \quad . \quad . \quad . \quad . \quad (14)$$

With this value of K

$$R = Mg\frac{3b^2 + k^2}{b^2 + k^2}\cos\theta + M(a - b)n^2. \quad (15)$$

If there were some arrangement to prevent slipping so long as the rolling body is in contact with the cylinder, then the condition necessary to ensure that the body should roll completely round the fixed cylinder is that R should always be positive. Now R has its least value when $\theta = \pi$, and if R is positive in this position, then it will be positive in all positions. Thus the body will make complete revolutions provided.

$$(a - b)n^2 > g\frac{3b^2 + k^2}{b^2 + k^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

But if no arrangement exists for preventing slipping, then slipping will occur when the expression for F given by (13) is greater than the expression for R multiplied by the coefficient of friction μ. That is, the purely rolling motion breaks down at the point where

$$\frac{g\sin\theta\dfrac{k^2}{b^2 + k^2}}{g\cos\theta\dfrac{3b^2 + k^2}{b^2 + k^2} + (a - b)n^2} = \mu \quad . \quad . \quad . \quad (17)$$

Suppose we take the value of n given by making the two sides of (16) equal. Then the point at which sliding begins is given by

$$\frac{k^2}{3b^2 + k^2} \cdot \frac{\sin\theta}{1 + \cos\theta} = \mu \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

or

$$\left.\begin{aligned}\tan\frac{\theta}{2} = \frac{\mu(3b^2 + k^2)}{k^2} &= \tfrac{17}{2}\mu \text{ for a sphere} \\ &= 7\mu \text{ for a cylinder}\end{aligned}\right\} \quad . \quad . \quad (19)$$

427. *A sphere of radius* a, *whose centre of mass is at a distance* b *from its centre, rolls without sliding down an inclined plane, the motion being such that the centre of mass moves in one vertical plane. To find the angular velocity after the body has turned through an angle θ, assuming that it started from rest with its centre of mass at its minimum distance from the plane.*

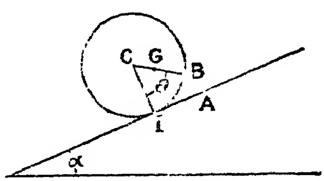


FIG. 186.

At the beginning of the motion B was in contact with A. The point of contact, I, at any instant, is the instantaneous centre of rotation. If ω is the angular velocity of the sphere, the linear velocity of G, the centre of mass, is therefore IG . ω.

But $$\mathrm{IG}^2 = \mathrm{IC}^2 + \mathrm{CG}^2 - 2\mathrm{IC}\,.\,\mathrm{CG}\cos\theta$$
$$= a^2 + b^2 - 2ab\cos\theta \quad . \quad . \quad . \quad . \quad (1)$$

If k is the radius of gyration about an axis through G perpendicular to the plane of motion, the whole kinetic energy of the body is

$$\tfrac{1}{2}\mathrm{M}(a^2 + b^2 - 2ab\cos\theta)\omega^2 + \tfrac{1}{2}\mathrm{M}k^2\omega^2 \quad . \quad . \quad . \quad (2)$$

In the position shown in the figure G is higher than I by an amount

$$a\cos\alpha - b\cos(\theta + \alpha) \quad . \quad . \quad . \quad . \quad (3)$$

At the beginning of the motion, when B was in contact with A, G was higher than I by

$$\mathrm{IA}\sin\alpha + (a - b)\cos\alpha = a\theta\sin\alpha + (a - b)\cos\alpha \quad . \quad (4)$$

Hence the work done by the weight while the body falls to its present position is

$$\mathrm{M}g\{a\theta\sin\alpha - b\cos\alpha + b\cos(\theta + \alpha)\} \quad . \quad . \quad . \quad (5)$$

The energy equation is therefore

$$\tfrac{1}{2}\mathrm{M}(a^2 + b^2 - 2ab\cos\theta + k^2)\omega^2$$
$$= \mathrm{M}g\{a\theta\sin\alpha - b\cos\alpha + b\cos(\theta + \alpha)\} \quad (6)$$

The kinetic energy can never be negative, and consequently the right-hand side of (6) can never be negative. In order, therefore, that the body should make complete revolutions, the right-hand side of (6) must be positive for all positive values of θ; that is, the minimum value of the right-hand side must be positive. It is clear that the sphere will not make a whole revolution except for particular values of a, b, and α. For example, if α is zero, it will not move at all, and if α is very small, the body will make very small oscillations.

Suppose that $\alpha = 20^\circ$ and $b = \frac{1}{2}a$. We will find whether the sphere started in the way we have supposed will make complete revolutions.

Let $$y = a\theta\sin\alpha - b\cos\alpha + b\cos(\theta + \alpha)$$
$$= b\{2\theta\sin\alpha - \cos\alpha + \cos(\theta + \alpha)\} \quad . \quad . \quad . \quad (7)$$

When y is a minimum

$$\frac{dy}{d\theta} = b\{2\sin\alpha - \sin(\theta + \alpha)\} = 0 \quad . \quad . \quad . \quad . \quad (8)$$

and $$\frac{d^2y}{d\theta^2} = -b\cos(\theta + \alpha) = \text{a positive quantity} \quad . \quad (9)$$

Equation (8) gives

$$\sin(\theta + \alpha) = 2\sin 20^\circ = 0{\cdot}6840 \quad . \quad . \quad . \quad . \quad (10)$$

The smallest root of this which makes $\cos(\theta + \alpha)$ negative is

$$\theta + \alpha = \frac{136^\circ\ 50\tfrac{1}{2}'}{180^\circ} \times \pi \text{ radians} \quad . \quad . \quad . \quad . \quad (11)$$

$$\theta = \frac{116^\circ\ 50\tfrac{1}{2}'}{180^\circ} \times \pi = 2{\cdot}030 \quad . \quad . \quad . \quad (12)$$

The minimum value of y is therefore

$$b\{4{\cdot}078 \sin 20° - \cos 20° + \cos 136° \, 50\tfrac{1}{2}'\} = -0{\cdot}275b \quad (13)$$

Thus y becomes negative before the body has turned through two right angles. Consequently y, and therefore ω, becomes zero before the body has turned through two right angles. The body will not roll down the plane, but will oscillate through an angle less than $116° \, 50\frac{1}{2}'$. In order to find the exact angle at which the body comes to rest, it is necessary to find the value of θ which makes y zero. For this particular case θ is about $\frac{50}{180}\pi$. That is, the body comes to rest after it has turned through about $50°$.

428. *A truck runs on six equal wheels each of mass* m *and radius* a. M *is the mass of the truck and wheels together*, k *the radius of gyration of a wheel about its axis. To find the acceleration of the truck down an incline at an angle* α *to the horizontal, neglecting the loss of energy by friction at the bearings.*

Let v be the velocity of the truck at any instant, ω the angular velocity of a wheel. Then the kinetic energy of each wheel is

$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}mk^2\omega^2 \quad \ldots \quad (1)$$

The kinetic energy of the body of the truck is

$$\tfrac{1}{2}(M - 6m)v^2 \quad \ldots \quad (2)$$

When the truck has run a distance x from rest down the plane the energy equation gives

$$\tfrac{1}{2}(M - 6m)v^2 + 6(\tfrac{1}{2}mv^2 + \tfrac{1}{2}mk^2\omega^2) = Mgx \sin \alpha \quad . \quad (3)$$

Since $a\omega = v$ this gives

$$\tfrac{1}{2}v^2\left(M + 6m\frac{k^2}{a^2}\right) = Mgx \sin \alpha \quad \ldots \quad (4)$$

Differentiating this with respect to x

$$v\frac{dv}{dx}\left(M + 6m\frac{k^2}{a^2}\right) = Mg \sin \alpha \quad \ldots \quad (5)$$

But
$$v\frac{dv}{dx} = \frac{dv}{dt} = \text{the acceleration}$$

Hence the acceleration down the plane is

$$\frac{dv}{dt} = \frac{Mg \sin \alpha}{M + 6m\frac{k^2}{a^2}} \quad \ldots \quad (6)$$

429. *A reel of cotton of mass* M *is placed on a rough inclined plane with its axis horizontal. The cotton unwinds from the upper side of the*

reel and runs parallel to the plane to a small light pulley over which it passes. The free end is attached to a mass m, *which falls vertically. To find the acceleration of the reel up the plane, assuming that it does not slip on the plane.*

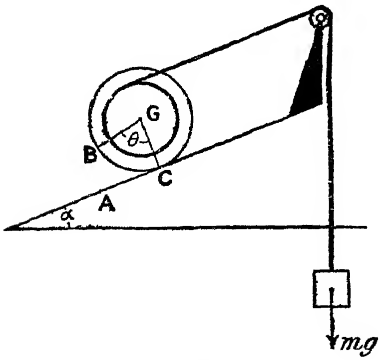


FIG. 187

Let a be the radius of tne rims of the reel, b the radius of the cylinder from which the cotton unwinds, k the radius of gyration of the reel. Let x be the distance which the reel has moved up the plane from rest, z the corresponding drop of m, and θ the angle through which the reel has turned. Let v denote $\frac{dx}{dt}$, the velocity of the reel.

The vertical rise of the centre of gravity of the reel is $x \sin \alpha$. Hence the work done on the system by the weights of the reel and the mass m is

$$mgz - \mathrm{M}gx \sin \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The energy equation is

$$\tfrac{1}{2}\mathrm{M}v^2 + \tfrac{1}{2}\mathrm{M}k^2\left(\frac{d\theta}{dt}\right)^2 + \tfrac{1}{2}m\left(\frac{dz}{dt}\right)^2 = mgz - \mathrm{M}gx \sin \alpha \quad . \quad (2)$$

Now the length of string that has passed over the pulley is x on account of the displacement of G and $b\theta$ on account of the rotation. Hence

$$z = x + b\theta = x + \frac{b}{a}x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and therefore
$$\frac{dz}{dt} = \left(1 + \frac{b}{a}\right)\frac{dx}{dt} = \left(1 + \frac{b}{a}\right)v \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The energy equation becomes, on expressing $\frac{d\theta}{dt}$ and $\frac{dz}{dt}$ in terms of v,

$$\tfrac{1}{2}\mathrm{M}v^2 + \tfrac{1}{2}\mathrm{M}\frac{k^2}{a^2}v^2 + \tfrac{1}{2}m\left(1 + \frac{b}{a}\right)^2 v^2 = gx\left\{m\left(1 + \frac{b}{a}\right) - \mathrm{M} \sin \alpha\right\} \quad (5)$$

Hence
$$\tfrac{1}{2}v^2 = \frac{agx\{m(a + b) - \mathrm{M}a \sin \alpha\}}{\mathrm{M}(a^2 + k^2) + m(a + b)^2} \quad . \quad . \quad . \quad . \quad (6)$$

The acceleration of the reel up the plane is

$$\frac{dv}{dt} = \frac{d}{dx}(\tfrac{1}{2}v^2) = \frac{ag\{m(a + b) - \mathrm{M}a \sin \alpha\}}{\mathrm{M}(a^2 + k^2) + m(a + b)^2} \qquad (7)$$

The expression for $\frac{dv}{dt}$ is negative if

$$Ma \sin \alpha > m(a + b) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

In this case the reel will roll down the plane and pull the mass m upwards.

430. *To show that when a body, such as a railway carriage, enters upon a curved path after travelling in a straight line, its velocity decreases if no forces act on it except the pressure of the rails, assuming the body remains straight.*

Suppose the curved path is a portion of a circle of radius r having the straight path as tangent. Let v_0 be the velocity on the straight path, v the velocity on the curved path.

The reactions of the rails do no work on the carriage. Consequently the kinetic remains unaltered on entering the curve. If k is the radius of gyration of the carriage about the vertical axis through its centre of mass, and ω the angular velocity, the equivalence of the kinetic energies gives

$$\tfrac{1}{2}Mv^2 + \tfrac{1}{2}Mk^2\omega^2 = \tfrac{1}{2}Mv_0^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

But clearly

$$r\omega = v \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence from (1) and (2)

$$v^2 + \frac{k^2}{r^2}v^2 = v_0^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

or

$$v^2 = \frac{r^2}{r^2 + k^2}v_0^2 \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If the curved path is not a circle we need only replace r by the radius of curvature of the curve at the point where the carriage is at the instant considered.

For any curves on a railway line r^2 is large compared with k^2, and consequently there is no appreciable alteration in the velocity.

431. By the method of Arts. 419 and 420, the principle of energy can easily be extended to motion in three dimensions. It can be proved that the kinetic energy of a body moving in any manner whatever is equal to the work done on the body since it was at rest.

When a nut moves along a bolt, or a bullet along the barrel of a rifled gun, the body has a rotation about the line along which the centre of gravity moves. If v is the velocity of translation along the axis, and ω the angular velocity, then the square of the resultant velocity of a particle at distance r from the axis is $v^2 + r^2\omega^2$. Hence the total kinetic energy of the body is

$$\tfrac{1}{2}\Sigma m(v^2 + r^2\omega^2) = \tfrac{1}{2}Mv^2 + \tfrac{1}{2}I\omega^2 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where I is the moment of inertia about the axis of rotation.

If p is the pitch of the screw along which the body moves, then since the body moves forward a distance p while it turns through 2π radians

$$v = \frac{\omega}{2\pi}p \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence the kinetic energy can be written

$$\tfrac{1}{2}\Big(M\frac{p^2}{4\pi^2} + I\Big)\omega^2 \text{ or } \tfrac{1}{2}\Big(M + \frac{4\pi^2}{p^2 I}\Big)v^2 \quad . \quad . \quad . \quad . \quad (3)$$

If a nut slides down a vertical bolt under its own weight, its kinetic energy when it has fallen a distance x, assuming that friction is negligible, is given by

$$\tfrac{1}{2}\Big(M + \frac{4\pi^2}{p^2}I\Big)v^2 = Mgx \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where $v = \frac{dx}{dt}\cdot$ On differentiating both sides with respect to x, we get

$$\Big(M + \frac{4\pi^2}{p^2}I\Big)\frac{dv}{dt} = Mg \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

which shows that the acceleration is constant and equal to

$$\frac{Mg}{M + \frac{4\pi^2}{p^2}I} = \frac{g}{1 + 4\pi^2\frac{k^2}{p^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

k being the radius of gyration about the axis of rotation.

We will work one more example on motion in three dimensions.

A solid cone is placed with a generating line in contact with a horizontal line on a rough inclined plane. Assuming that no sliding takes place, to find the angular velocity when the line of contact is a line of greatest slope.

Let r and h be the radius of the base and the height of the cone, M its mass, I its moment of inertia about a generating line of the cone. Let 2α be the vertical angle of the cone.

The moment of inertia of the cone about its axis is, by Art. 203, Ex. 3*b*,

$$\tfrac{3}{10}Mr^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and about an axis through its vertex perpendicular to this, its moment of inertia is, by Art. 207, Ex. 2,

$$\tfrac{3}{5}(h^2 + \tfrac{1}{4}r^2)M \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Hence, by Art. 209, the moment of inertia I is

$$I = \tfrac{3}{10}Mr^2\cos^2\alpha + \tfrac{3}{5}M(h^2 + \tfrac{1}{4}r^2)\sin^2\alpha \quad . \quad . \quad . \quad (9)$$

If ω is the angular velocity required, the kinetic energy is $\frac{1}{2}I\omega^2$, since the generator in contact with the plane is the instantaneous axis of rotation.

Now as the cone rolls about on the plane keeping its vertex fixed, its centre of gravity describes a circle in a plane parallel to the inclined plane, and the radius of this circle is $\frac{3}{4}h\cos\alpha$. In the initial position the radius which passes through the centre of gravity is horizontal, and in the final position it makes the same angle with the horizontal as the inclined plane itself makes. If β is the inclination of the plane, it

CHAPTER XXI

MOTION OF A RIGID BODY ABOUT A FIXED AXIS AND SMALL OSCILLATIONS OF A RIGID BODY

432. Motion about a Fixed Axis.—The problem of motion about a fixed axis can be solved by using the equations of motion given in Art. 406. But if we use these equations we shall have to eliminate the two unknown components of the reaction of the axis on the body. If, however, these forces are not required, it is much easier to find the motion by the method given here.

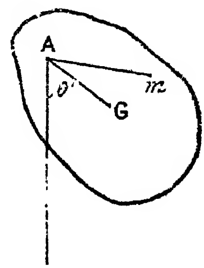


FIG. 188.

Let ω denote the angular velocity. Let r be the distance from the axis of any particle m of the rigid body. The momentum of this particle is $mr\omega$. Therefore its moment of momentum is $mr^2\omega$. Now by Art. 348

$$\frac{d}{dt}(mr^2\omega) = \text{moment, about fixed axis, of all forces acting on } m \quad . \quad (1)$$

But since r is independent of the time the left-hand side of this equation is

$$mr^2\frac{d\omega}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and $\frac{d\omega}{dt}$ is the same for every particle of the body.

By summing both sides of such equations as (1) for every particle of the body, we get

$$\frac{d\omega}{dt}\Sigma mr^2 = \text{sum of moments about the axis of all the forces acting on the particles} \quad . \quad . \quad . \quad . \quad (3)$$

Now the right-hand side of (1) contains the moments of the actions on m of the neighbouring particles. But the terms corresponding to these mutual actions will destroy each other in the sum because action and reaction occur in the sum, and these have equal but opposite moments about any point. Consequently the right-hand side of (3) is merely the moment about the fixed axis of the external forces on the rigid body.

Now if AG $= h$, and k is the radius of gyration of the body about the axis through the centre of mass G parallel to the fixed axis, then

$$\Sigma mr^2 = Mk^2 + Mh^2 \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

by Art. 205.

Hence equation (3) becomes

$$M(k^2 + h^2)\frac{d\omega}{dt} = N \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

where N denotes the moment of the external forces about the fixed axis, this moment being considered positive when it increases ω.

If a plane fixed in the body and containing the axis, say the plane through G, makes an angle θ with a plane fixed in space and also containing the axis, then

$$\omega = \frac{d\theta}{dt}, \quad \frac{d\omega}{dt} = \omega\frac{d\omega}{d\theta} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

and therefore
$$M(k^2 + h^2)\frac{d^2\theta}{dt^2} = N \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

becomes
$$M(k^2 + h^2)\omega\frac{d\omega}{d\theta} = N \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

Equation (3) above has exactly the same form as equation (3) of Art. 406, since Σmr^2 is the moment of inertia about the fixed axis. In this case, however, moments are taken about a fixed axis, whereas in Art. 406 moments were taken about an axis through the centre of mass.

433. Motion of a Rigid Body under Gravity about a Fixed Horizontal Axis.—Let A be the fixed axis, G the centre of gravity, θ the angle which AG makes with the vertical as shown in Fig. 188. Let AG $= h$. The only force which has a moment about A is the weight Mg. Hence the equation of motion is

$$M(k^2 + h^2)\frac{d^2\theta}{dt^2} = -Mgh \sin\theta \quad . \quad . \quad . \quad . \qquad (1)$$

or
$$\frac{k^2 + h^2}{h} \cdot \frac{d^2\theta}{dt^2} = -g \sin\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

Now the equation of motion of a particle at the end of a string of length l is, by Art. 330,

$$l\frac{d^2\theta}{dt^2} = -g \sin\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

Equations (2) and (3) will be identical if

$$\frac{k^2 + h^2}{h} = l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

Consequently, if θ and $\frac{d\theta}{dt}$ are the same for the rigid body and the string of length l given by (4), then since their angular accelerations are

always equal, they will always move in unison. If one of them is making small oscillations in a period τ, the other will be making small oscillations in the same period; if one is making complete revolutions the other will be making complete revolutions; also AG and the pendulum-string will always make the same angle with the vertical.

Since the period of a small oscillation of a simple pendulum is approximately

$$2\pi\sqrt{\frac{l}{g}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

it follows that the period of a small oscillation of the rigid body will be approximately

$$2\pi\sqrt{\frac{k^2+h^2}{hg}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

It should be borne in mind that, although these expressions for the periods of oscillation are only approximate the error is the same in both if the angular amplitudes are the same. Even for large amplitudes, when the preceding expressions are considerably in error, the actual periods of oscillation are the same for the same amplitudes.

Writing $\omega\dfrac{d\omega}{d\theta}$ for $\dfrac{d^2\theta}{dt^2}$ and integrating (2), we get

$$\tfrac{1}{2}\frac{k^2+h^2}{h}\omega^2 = g\cos\theta + \mathrm{C} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

This could have been written down immediately from the energy principle.

Suppose $\omega = \omega_0$ when $\theta = 0$. Then

$$\tfrac{1}{2}\frac{k^2+h^2}{h}\omega_0^2 = g + \mathrm{C} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Therefore, by subtracting this from (7)

$$\tfrac{1}{2}\frac{k^2+h^2}{h}(\omega^2-\omega_0^2) = -g(1-\cos\theta) \quad . \quad . \quad . \quad (9)$$

In the highest position, when $\theta = \pi$,

$$\omega^2 = \omega_0^2 - \frac{4hg}{k^2+h^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

If this expression for ω^2 is negative the interpretation is that the body cannot reach the highest position. Consequently, the condition that the body should make complete revolutions is that

$$\omega_0^2 > \frac{4hg}{k^2+h^2}$$

The component actions of the axis on the body along and perpendicular to GA can be found by the method of Art. 417.

434. Compound Pendulum.—A rigid body mounted on a horizontal axis and making small oscillations under gravity is called a *compound*

pendulum. It was shown in the last article that it oscillates in the same time as a simple pendulum of length l, where

$$l = \frac{k^2 + h^2}{h} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The length l is called the *length of the simple equivalent pendulum.* A better name would be the *length of the equivalent simple pendulum*, but the other name has become established.

Since k is the radius of gyration about the axis through the centre of mass parallel to the fixed axis, it follows that k is the same for all axes which have the same direction in the body. Consequently, l is the same for all axes lying on the surface of a cylinder having a line through the centre of mass as axis of symmetry. That is, if the body be allowed to oscillate in turn about a number of parallel axes in the body the time of oscillation will be the same about all the axes at the same distance from the centre of mass.

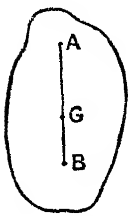


FIG. 189.

435. Centres of Suspension and Oscillation.—If, in a compound pendulum, the perpendicular GA on the axis of suspension be produced backwards to B so that

$$AB = l = \frac{k^2 + h^2}{h} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

then A and B are called the centres of suspension and oscillation respectively.

Now suppose l is given as well as k, and we want to find h. From (1)

$$h^2 - hl + k^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This equation shows that there are two values of h for which the time of oscillation is the same. Let h_1 and h_2 be these values. Then

$$h_1 + h_2 = l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$h_1 h_2 = k^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If AG is h_1 then BG is h_2, as equations (3) and (1) show. The length of BG, from (4), is

$$h_2 = \frac{k^2}{h_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

From the symmetry of the relations (3) and (4), it follows that the centres of suspension and oscillation are interchangeable. If the body were suspended from an axis through B parallel to the one through A, the body would therefore oscillate in the same time.

The centre of oscillation of a compound pendulum will have exactly the same motion as the bob of the equivalent simple pendulum provided they are both started in the same way.

For different axes of suspension parallel to a given axis through the centre of mass there is a minimum value of l, and therefore a minimum value of the period of oscillation. This minimum value of l is found

by varying h in (1) and keeping k constant. We may use the calculus to get the minimum or the following algebraical method :—

$$l^2 = \left(\frac{k^2}{h} + h\right)^2 = \left(\frac{k^2}{h} - h\right)^2 + 4k^2 \quad . \quad . \quad . \quad . \quad (6)$$

For variations in h, l^2 obviously has its least value when $h = k$, for then the variable part of l^2 has its least value, which is zero. Hence the minimum value of l is $2k$. When l has this value h_1 and h_2 are equal.

436. The Method of finding g by a Pendulum.—A very fair value of g can be obtained by observations on a pendulum formed by a heavy metal bob with a diameter of about half an inch at the end of a piece of thread about four feet long. This may be regarded as a simple pendulum. But it is not a theoretical simple pendulum, for all its mass is not concentrated at one point. The ideal simple pendulum is not realisable, and even in the best approach to it the mass of the string is not zero, and we do not know exactly where the centre of mass of the bob is. To avoid all these difficulties when an accurate value of g is wanted, it is found best to use a heavy rigid body as a pendulum and find two points A and B on opposite sides of the centre of gravity (and such that AG and BG are not equal) about each of which the body makes a complete oscillation in the same given period of time—usually about two seconds. Then the distance AB is measured and the value of g is given by the equation

$$\tau = 2\pi\sqrt{\frac{AB}{g}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or

$$g = \frac{4\pi^2 . AB}{\tau^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This was the method used by Captain Kater about 1818 to determine g at London and other places in England.

437. Time of Oscillation of Certain Bodies.

EXAMPLE 1.—*Assuming that a sphere at the end of a string oscillates as if it were rigidly attached to the axis, to find the time of oscillation.*

Let a be the radius of the sphere and h the distance from the point of suspension to the centre of the sphere. Then the length of the equivalent simple pendulum is

$$\frac{\frac{2}{5}a^2}{h} + h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The time of oscillation is therefore

$$\tau = 2\pi\sqrt{\frac{h + \frac{2a^2}{5h}}{g}} = 2\pi\sqrt{\frac{h}{g}}\left(1 + \frac{a^2}{5h^2}\right) \quad . \quad . \quad (2)$$

nearly, provided a is small compared with h.

Suppose $h = 50$ inches, $a = \frac{1}{2}$ inch. Then

$$\tau = 2\pi\sqrt{\frac{h}{g}}\left(1 + \frac{1}{5 \times 10^4}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The error in regarding the system as a simple pendulum is thus only $\frac{1}{5 \times 10^4}$ of the whole time of oscillation.

EXAMPLE 2.—*Time of oscillation of a solid cylindrical rod of length* b *and radius* r *about an axis perpendicular to its axis of symmetry and at a distance* c *from its centre of mass.*

The moment of inertia about the axis through the centre of mass parallel to the fixed axis is, by Art. 207, Ex. 1,

$$M\left(\frac{b^2}{12} + \frac{r^2}{4}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The length of the equivalent simple pendulum is, therefore,

$$l = c + \frac{b^2 + 3r^2}{12c} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The time of oscillation is thus

$$\tau = 2\pi\sqrt{\frac{c + \frac{b^2 + 3r^2}{12c}}{g}} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The minimum time of oscillation for all axes parallel to the given axis is

$$\tau = 2\pi\sqrt[4]{\frac{b^2 + 3r^2}{3g^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

since the minimum value of l is $2k$.

For a thin rod r may usually be neglected.

The minimum time of oscillation of a yard-stick whose diameter is half an inch is

$$\tau = 2\pi\sqrt[4]{\frac{9 + \frac{3}{48^2}}{3 \times 32{\cdot}2^2}} = 1{\cdot}45730 \text{ secs.} \quad . \quad . \quad . \quad (8)$$

If r be neglected this result will be diminished by only 5 in the last place.

EXAMPLE 3.—*A thin uniform rod of length* b *is suspended by two light strings each of length* $\frac{\sqrt{5}}{4}$ b, *which join the ends to the same fixed point. To find the periods of oscillation of the rod in the plane containing the strings and perpendicular to this plane.*

The distance of the point of suspension from the centre of mass of the rod is $\frac{1}{4}b$. For motion in the plane of the strings the axis is perpendicular to the rod, and consequently $k^2 = \frac{1}{12}b^2$. The length of the equivalent simple pendulum is, therefore,

$$l = \frac{\frac{1}{12}b^2}{\frac{1}{4}b} + \frac{1}{4}b = \frac{7}{12}b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The time of oscillation is

$$\tau = 2\pi\sqrt{\frac{7b}{12g}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

When the rod swings perpendicular to the plane of the strings, since the thickness is negligible, $k^2 = 0$ because the axis for k is along the rod. In this case, therefore,

$$l = \tfrac{1}{4}b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and

$$\tau = 2\pi\sqrt{\frac{b}{4g}}.$$

438. Axes in a Rigid Body which may be regarded as Fixed Axes in taking Moments.

If ω is the angular velocity of a rigid body, I the moment of inertia about any axis of the body perpendicular to the plane of motion, N the moment of the forces about the same axis, we have already found that an equation of the form

$$I\frac{d\omega}{dt} = N \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

is true in two cases, namely, when the axis is fixed in space, and when the axis passes through the centre of mass. We shall now examine whether this equation is true for any other axes in the body.

Let OX and OY be a pair of axes fixed in space in a plane parallel to the motion. Let C represent an axis fixed in the body and perpendicular to the plane XOY. Let P be the position of any particle m of the body. Let the co-ordinates of C be ξ, η, and let CP $= r$. Then the co-ordinates of P are

$$\left.\begin{aligned} x &= \xi + r\cos\theta \\ y &= \eta + r\sin\theta \end{aligned}\right\} \quad (2)$$

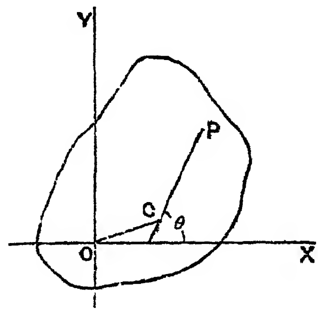


FIG. 190.

We shall write α and β for the accelerations of C parallel to the fixed axes. That is

$$\alpha = \frac{d^2\xi}{dt^2}, \quad \beta = \frac{d^2\eta}{dt^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If N is the moment about O of the external forces on the body we know that

$$\Sigma m\left(x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2}\right) = N \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

for this equation is obtained by summing such equations as (7) of Art. 348 for every particle.

Now

$$x\frac{d^2y}{dt^2} = (\xi + r\cos\theta)\left(r\cos\theta\frac{d^2\theta}{dt^2} - r\sin\theta\left|\frac{d\theta}{dt}\right|^2 + \frac{d^2\eta}{dt^2}\right) \quad . \quad (5)$$

$$= (\xi + r\cos\theta)\left(r\cos\theta\frac{d\omega}{dt} - r\sin\theta\,.\,\omega^2 + \beta\right) \quad . \quad . \quad (6)$$

Now the origin O may be taken at any *fixed* point we choose. Let it be taken, then, at the instantaneous position of C. In this case

$$\xi = 0, \eta = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Therefore, since α and β remain unaffected,

$$x\frac{d^2y}{dt^2} = r^2\cos^2\theta\frac{d\omega}{dt} - r^2\sin\theta\cos\theta\,.\,\omega^2 + r\cos\theta\,.\,\beta \quad . \quad (8)$$

Similarly

$$y\frac{d^2x}{dt^2} = -r^2\sin^2\theta\frac{d\omega}{dt} - r^2\sin\theta\cos\theta\,.\,\omega^2 + r\sin\theta\,.\,\alpha \quad (9)$$

Hence $$x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = r^2\frac{d\omega}{dt} + r\cos\theta\,.\,\beta - r\sin\theta\,.\,\alpha \quad . \quad . \quad (10)$$

Thus equation (4) becomes

$$\Sigma m\left(r^2\frac{d\omega}{dt} + r\cos\theta\,.\,\beta - r\sin\theta\,.\,\alpha\right) = \mathrm{N} \quad . \quad . \quad . \quad (11)$$

or, if I denotes the moment of inertia about C,

$$\mathrm{I}\frac{d\omega}{dt} + \beta\Sigma mr\cos\theta - \alpha\Sigma mr\sin\theta = \mathrm{N} \quad . \quad . \quad . \quad (12)$$

This equation will be of the same form as (1) provided

$$\beta\Sigma mr\cos\theta - \alpha\Sigma mr\sin\theta = 0 \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Let $\bar{x}$ and $\bar{y}$ be the co-ordinates of the centre of mass referred to the axes through C. Then

$$\Sigma mr\cos\theta = \bar{x}\mathrm{M} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

$$\Sigma mr\sin\theta = \bar{y}\mathrm{M} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

The condition (13) can therefore be written

$$\beta\bar{x} - \alpha\bar{y} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

If C is itself the centre of mass of the body then $\bar{x}$ and $\bar{y}$ are each zero, and therefore equation (1) is true. This verifies our previous work.

In general, if

$$\frac{\alpha}{\beta} = \frac{\bar{x}}{\bar{y}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

equation (1) is true for the axis through C. Now if (17) is true, the acceleration of C is in the line joining C to the centre of mass. It

follows, then, that we may take moments about any point moving with a rigid body as if it were a fixed point, provided the acceleration of this point is along the line joining it to the centre of mass.

439. The point of contact of a solid of revolution, such as a sphere or cylinder, rolling without slipping on a fixed surface, may be regarded as a fixed point in taking moments.

To prove this it is only necessary to show that the acceleration of this point of contact is along the line joining it to the centre of mass.

Let ω be the angular velocity and a the radius of the solid. The point of contact is the instantaneous centre of rotation, and therefore the velocity of the centre of mass is $a\omega$. The component acceleration of the centre of mass in its direction of motion is thus

$$a\frac{d\omega}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

It has another component perpendicular to this whose magnitude we do not need.

Now the acceleration of the instantaneous centre is its acceleration relative to G together with the acceleration of G. But relative to G the body is rotating with an angular velocity ω. Consequently the instantaneous centre has a relative acceleration

$$a\frac{d\omega}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in the direction opposite to (1), and $a\omega^2$ perpendicular to this. Hence the true acceleration of the instantaneous centre has no component perpendicular to the line joining it to G. Its resultant acceleration is therefore along this line. Thus the theorem is proved.

440. *A sphere rolls on the outer surface of a rough fixed sphere. If the rolling sphere started from the highest position with a very small velocity, to find the subsequent motion.*

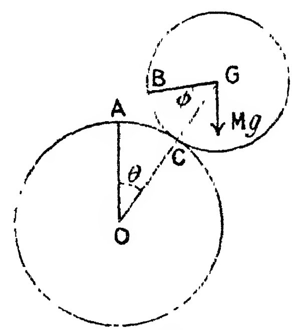


FIG. 191.

Let a and b denote the radii of the rolling and fixed spheres. A is the highest point of the fixed sphere, B the point of the rolling sphere which was originally in contact with A; C is the instantaneous centre of rotation of the rolling sphere. Let ω be the angular velocity of the rolling sphere. By the last article we may take moments about C as long as there is no slipping. If M denotes the mass and k the radius of gyration about an axis through G, we get, on taking moments,

$$M(a^2 + k^2)\frac{d\omega}{dt} = Mga \sin\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now

$$b\theta = a\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence the angle turned through by the rolling body is

$$\theta + \phi = \theta\left(1 + \frac{b}{a}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Therefore
$$\omega = \left(1 + \frac{b}{a}\right)\frac{d\theta}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Equations (1) and (4) give

$$\left(1 + \frac{b}{a}\right)(a^2 + k^2)\frac{d^2\theta}{dt^2} = ag \sin\theta \quad . \quad . \quad . \quad . \quad (5)$$

Now, since
$$\frac{d^2\theta}{dt^2} = \frac{1}{2}\cdot\frac{d}{d\theta}\left\{\left(\frac{d\theta}{dt}\right)^2\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

we find on integrating (5) with respect to θ,

$$\frac{1}{2}\left(1 + \frac{b}{a}\right)(a^2 + k^2)\left(\frac{d\theta}{dt}\right)^2 = -ag\cos\theta + C \quad . \quad . \quad (7)$$

Now the velocity at the highest position is supposed to be small enough to be negligible. We are supposing the body to have been disturbed from the position of unstable equilibrium. Taking the velocity to be zero when θ was zero, (7) gives

$$0 = -ag + C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Subtracting this from (7)

$$\frac{1}{2}\left(1 + \frac{b}{a}\right)(a^2 + k^2)\left(\frac{d\theta}{dt}\right)^2 = ag(1 - \cos\theta)$$

$$= ag\,.\,2\sin^2\frac{\theta}{2} \quad . \quad . \quad . \quad . \quad (9)$$

Hence
$$\frac{d\theta}{dt} = 2a\sqrt{\frac{g}{(a+b)(a^2+k^2)}}\sin\frac{\theta}{2}$$

$$= 2n\sin\frac{\theta}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where
$$n = a\sqrt{\frac{g}{(a+b)(a^2+k^2)}} = \sqrt{\frac{5g}{7(a+b)}} \quad . \quad . \quad . \quad (11)$$

for a sphere, since $k^2 = \frac{2}{5}a^2$.

Therefore
$$n\frac{dt}{d\theta} = \frac{1}{2\sin\frac{\theta}{2}} = \frac{1}{4\sin\frac{\theta}{4}\cos\frac{\theta}{4}} = \frac{\frac{1}{4}\sec^2\frac{\theta}{4}}{\tan\frac{\theta}{4}} \quad . \quad . \quad . \quad (12)$$

Hence
$$nt + \text{a constant} = \log_e \tan\frac{\theta}{4} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

or
$$\tan\frac{\theta}{4} = Ae^{nt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

If we attempt to make θ zero when t is zero, we shall find that A is zero. This is a consequence of assuming the velocity zero at the highest position. It tells us, in fact, that a body cannot leave a position of equilibrium, even if it is unstable, unless it has some velocity. Yet although the body could not have acquired the velocity given by equation (10) by falling from the highest position, it could nevertheless be started from any other position with the velocity which it would have in that position according to our equations. Then our results would be absolutely correct for the subsequent motion. Suppose that $t = 0$ when $\theta = \alpha$. Then adjusting the constant in (14) to give this condition we find

$$\tan\frac{\theta}{4} = e^{nt}\tan\frac{\alpha}{4} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

If R and F are the normal force and the friction acting on the rolling body, we find, on resolving along and perpendicular to OG for the motion of the centre of mass,

$$\begin{aligned} R &= Mg\cos\theta - M(a+b)\left(\frac{d\theta}{dt}\right)^2 \\ &= Mg\cos\theta - Mg\frac{2a^2}{a^2+k^2}(1-\cos\theta) \\ &= \frac{Mg}{a^2+k^2}\{(3a^2+k^2)\cos\theta - 2a^2\} \quad . \quad . \quad . \quad . \quad (16) \end{aligned}$$

and

$$\begin{aligned} F &= Mg\sin\theta - M(a+b)\frac{d^2\theta}{dt^2} \\ &= \frac{Mg\sin\theta}{a^2+k^2}\{(a^2+k^2) - a^2\} \\ &= \frac{k^2}{a^2+k^2}Mg\sin\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17) \end{aligned}$$

If there is no possibility of slipping, the rolling sphere will leave the fixed one at the point where R becomes zero, that is, where

$$\cos\theta = \frac{2a^2}{3a^2+k^2} = \frac{2a^2}{3a^2+\frac{2}{5}a^2} = \frac{10}{17} = 0{\cdot}5882 \quad . \quad (18)$$

But if there is no arrangement to prevent slipping, the moving sphere will begin to slide at the point where $\frac{F}{R}$ is equal to the coefficient of friction. Suppose $\mu = 0{\cdot}4$. Then slipping will occur where

$$(0{\cdot}4)(\tfrac{17}{5}a^2\cos\theta - 2a^2) = \tfrac{2}{5}a^2\sin\theta$$

or

$$17\cos\theta - 10 = 5\sin\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

The solution of this is $\cos\theta = 0{\cdot}7743 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$

On comparing this with the result of (18) it is evident that slipping begins before the point given by (18). In the second case the sphere

rolls as far as the point given by (20), then rolls and slides some distance further and finally leaves the fixed sphere. When sliding occurs there is less friction than there would be for pure rolling. Consequently the centre of mass will move a little faster than in the case of rolling without sliding, and therefore the rolling sphere will leave the fixed sphere somewhere between the points given by (18) and (20) and very much nearer the former position. The values of θ given by (18) and (20) are $53^\circ\ 58'$ and $39^\circ\ 16'$.

The point of separation can be found with as much accuracy as we desire by solving the equations of motion of the centre of mass with the condition that $F = \mu R$ from the point at which slipping begins. But as the work is laborious we shall omit it.

It should be noticed that all the equations as far as (18) are true only while there is no slipping.

441. Small Oscillations with One Degree of Freedom.—If a body is slightly displaced from a position of stable equilibrium, the forces brought into play are such as will bring it back to the equilibrium position. But if no frictional resistances act, the body will not be at rest when it arrives in the equilibrium position. The restoring forces will have done work on the body, and this work will appear as kinetic energy in the body. The body will therefore have a velocity which will carry it beyond the equilibrium position till the restoring forces (which always act towards the equilibrium position) have brought it to rest again. Then the body will fall back through the equilibrium position to its original starting-point, and so on for ever. Thus if no energy is destroyed by frictional forces, nor carried away by such things as air-waves, a body will oscillate about a position of stable equilibrium for ever.

If the body has only one degree of freedom, or if the motion is started in such a way that one co-ordinate fixes its position at any instant, the equations of motion will, in nearly every case, lead to an equation of the form

$$\frac{d^2\theta}{dt^2} = -k^2\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where θ is a co-ordinate which fixes the position of the body. This is exactly the same type of equation as we have already had for small oscillations of a particle as well as for the compound pendulum.

442. *A small body is suspended by a straight cylindrical wire with its centre of mass in the axis of the wire. The body is turned through an angle β about the axis of the wire and then let go; to find the motion and the time of a small oscillation.*

Let θ be the angle through which the body is turned from the equilibrium position at any instant, $c^2\theta$ the couple exerted by the wire when its lower end is turned through θ. Let I be the moment of inertia of the body about the axis of the wire.

The equation for the rotational motion is

$$I\frac{d^2\theta}{dt^2} = -c^2\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The solution of this is

$$\theta = \mathrm{A} \sin \frac{c}{\sqrt{\mathrm{I}}}t + \mathrm{B} \cos \frac{c}{\sqrt{\mathrm{I}}}t \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From the way in which the body was started

$$\left.\begin{aligned} \frac{d\theta}{dt} &= 0 \\ \theta &= \beta \end{aligned}\right\} \text{ when } t = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The first of these conditions gives

$$\mathrm{A} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and the second gives

$$\mathrm{B} = \beta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Hence
$$\theta = \beta \cos \frac{c}{\sqrt{\mathrm{I}}}t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The period of a complete oscillation is

$$\tau = 2\pi\frac{\sqrt{\mathrm{I}}}{c} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

If a is the radius and l the length of the wire, it is shown in Art. 224 that the couple required to twist the wire through θ radians is

$$\frac{\pi n\theta}{2l}a^4 \text{ foot-lbs.} = \frac{\pi n g a^4}{2l}\theta \text{ foot-poundals} \quad . \quad . \quad . \quad (8)$$

Hence
$$c^2 = \frac{\pi n g a^4}{2l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

From (7) and (9) we get $\tau^2 = \dfrac{8\pi l \mathrm{I}}{n g a^4}$ . . . . (10)

Whence
$$n = \frac{8\pi l \mathrm{I}}{g a^4 \tau^2} \text{ lbs. per square foot} \quad . \quad . \quad (11)$$

This gives n, the modulus of rigidity of the material of the wire, since all the quantities in the expression for n can be measured or observed.

443. *A circular hoop of radius* a *and mass* M *hangs from a ceiling by three strings each of length* l. *The hoop is twisted through a small angle* β *about its axis and then let go; to find the motion and period of oscillation.*

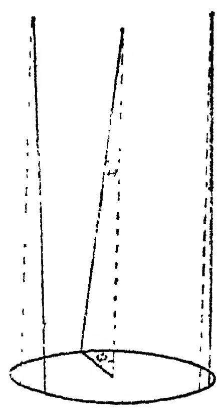


FIG. 192.

Let θ be the angle which the strings make with the vertical t seconds after the start. Let ϕ be the angle through which the hoop is displaced from the equilibrium position. T is the tension in one of the strings, S the sum of the three tensions. If the three points at which the strings are attached to the hoop lie at the corners of an equilateral triangle, T is one-third

of S, but in other cases this will not be true. The vertical displacement z of the centre of mass is

$$z = l(1 - \cos\theta) = l\frac{\theta^2}{2} \text{ nearly} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and since θ is always small, this is a small quantity compared with θ. The horizontal displacement of any particle of the hoop is nearly proportional to the first power of θ, and is consequently much greater than the vertical displacement. We may therefore neglect the vertical motion altogether.

Then resolving vertically

$$M\frac{d^2z}{dt^2} = S\cos\theta - Mg \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

that is,

$$0 = S - Mg \text{ nearly} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Now the horizontal forces cause the horizontal motion. The only horizontal forces are the horizontal components of the tensions. The horizontal component of the tension T is T sin θ, and it acts along a small chord of the hoop. Its moment about the axis of the hoop is therefore nearly aT sin θ, or, since θ is small, $aT\theta$. The sum of the moments of the horizontal components of the tensions is thus $aS\theta$. Hence the equation for rotational motion is

$$I\frac{d^2\phi}{dt^2} = -aS\theta = -aMg\theta \quad . \quad . \quad . \quad . \quad (4)$$

Now it is clear from the figure that

$$a\phi = l\theta \text{ nearly} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Consequently (4) becomes

$$I\frac{l}{a}\cdot\frac{d^2\theta}{dt^2} = -aMg\theta \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Whence

$$\frac{d^2\theta}{dt^2} = -\frac{Ma^2}{I}\cdot\frac{g}{l}\theta$$

$$= -\frac{g}{l}\theta \text{ for a hoop} \quad . \quad . \quad (7)$$

This is exactly the same type of equation as for the motion of a simple pendulum of length l. The solution is

$$\theta = A\cos\sqrt{\frac{g}{l}}t + B\sin\sqrt{\frac{g}{l}}t \quad . \quad . \quad . \quad . \quad (8)$$

And by the initial conditions, namely, that

$$\left.\begin{aligned} \frac{l}{a}\theta (= \phi) &= \beta \\ \frac{d\theta}{dt} &= 0 \end{aligned}\right\} \text{ when } t = 0$$

we get

$$\theta = \frac{a}{l}\beta\cos\sqrt{\frac{g}{l}}t \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The period of a small oscillation is

$$\tau = 2\pi\sqrt{\frac{l}{g}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

All the work of this article as far as equation (6) will apply equally well to any other body having its centre of mass in the vertical line through the centre of the circle which passes through the lower ends of the three strings. We need only substitute the proper value of I in equation (6), and then the rest of the work is similar. Thus for a uniform circular disc, the points of support being on the edge of the disc,

$$\left.\begin{aligned} I &= \tfrac{1}{2}Ma^2 \\ \tau &= 2\pi\sqrt{\frac{l}{2g}} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

If k is the radius of gyration of a body in the general case

$$\tau = 2\pi\sqrt{\frac{k^2 l}{a^2 g}} = 2\pi\frac{k}{a}\sqrt{\frac{l}{g}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

444. Rocking Bodies.

One body rests in equilibrium on the highest point of another body. The upper body is then rolled slightly from the equilibrium position. Assuming that the vertical sections of the two bodies containing all the points of contact are circles, to find the subsequent rolling motion as long as the displacement is small.

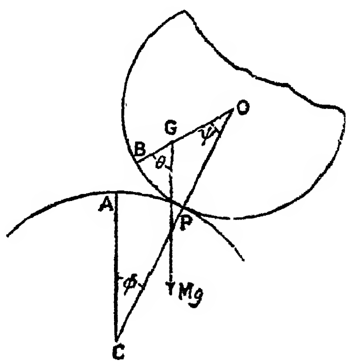


Fig. 193.

G is the centre of mass, and O the centre of the circular section of the rolling body; A is the highest point of the fixed body, B the point of the rolling body which was in contact with A in the equilibrium position; P is the point of contact when the rolling body has turned through θ from the equilibrium position; C is the centre of the section of the fixed body.

Let BG $= h$, BO $= r$, AC $=$ R.

When the vertical through G is between A and P, the perpendicular from P on this vertical is

$$\text{GO} \sin\theta - \text{PO}\sin\phi = (r-h)\sin\theta - r\sin\phi \quad . \quad . \quad (1)$$

Hence the moment about P of the weight in the direction turning the body back towards the equilibrium position is

$$Mg\{(r-h)\sin\theta - r\sin\phi\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Now on account of rolling

$$\text{arc AP} = \text{arc BP},$$

that is,
$$R\phi = r\psi = r(\theta - \phi) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Therefore
$$\phi = \frac{r}{R + r}\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Taking moments about P for the motion, we get

$$M(k^2 + h^2)\frac{d^2\theta}{dt^2} = -Mg\{(r - h)\sin\theta - r\sin\phi\} \quad . \quad (5)$$

where k is the radius of gyration of the rolling body about its centre of mass.

Now as long as the angles θ and ϕ are small we may replace the sines by the angles. Then the coefficient of $-Mg$ in (5) becomes

$$(r - h)\theta - r\phi = (r - h)\theta - \frac{r^2}{R + r}\theta$$

$$= \left(\frac{Rr}{R + r} - h\right)\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The equation of motion (5) now becomes, for small displacements,

$$M(k^2 + h^2)\frac{d^2\theta}{dt^2} = -Mg\left\{\frac{Rr}{R + r} - h\right\}\theta \quad . \quad . \quad . \quad (7)$$

This takes the well-known form

$$\frac{d^2\theta}{dt^2} = -c^2\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

provided
$$\frac{Rr}{R + r} > h$$

that is, provided
$$\frac{1}{h} > \frac{1}{R} + \frac{1}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The solution of (8) is $\theta = A\sin(ct + \beta)$ . . . . (10)
and the period of oscillation is

$$\tau = \frac{2\pi}{c} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

where
$$c^2 = \frac{g}{k^2 + h^2}\left(\frac{Rr}{R + r} - h\right) \quad . \quad . \quad . \quad . \quad . \quad (12)$$

But if
$$\frac{1}{h} < \frac{1}{R} + \frac{1}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

then the equation of motion (7) takes the form

$$\frac{d^2\theta}{dt^2} = n^2\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

the solution of which is, by Art. 337,

$$\theta = Ce^{nt} + De^{-nt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

In the general case, when C is not zero, θ becomes large after a short time, and when this happens our approximate equation is no longer of any use.

When the condition (9) holds, the body will oscillate about the equilibrium position. In this case, therefore, the equilibrium is clearly stable. But when the condition (13) is true, the body will leave the equilibrium position if it is slightly disturbed, and therefore the equilibrium is unstable.

It will be easily seen, by drawing other figures and following out the argument again, that if the concavity of either the moving or the fixed body is on the opposite side from that on which we have taken it, then the sign of the corresponding radius must be reversed. Also if either body is plane in the neighbourhood of the point of contact, the corresponding radius must be taken as infinite.

If the sections of the bodies in the plane of motion are not circles, it is only necessary to replace r and R by the radii of curvature of the sections. The errors due to taking the circles of curvature of the sections instead of the true curves will be small quantities of higher orders than those appearing in our equations, and will therefore be negligible.

445. We will apply the results of the last article to a few examples of rocking bodies.

EXAMPLE 1.—*A homogeneous solid hemisphere on a plane with its flat side uppermost.*

If r is the radius of the hemisphere,

$$h = \tfrac{5}{8}r, \quad R = \infty, \quad k^2 = \tfrac{2}{5}r^2 - (\tfrac{3}{8}r)^2,$$

$$k^2 + h^2 = \tfrac{2}{5}r^2 - (\tfrac{3}{8}r)^2 + (\tfrac{5}{8}r)^2 = \tfrac{13}{20}r^2,$$

$$\frac{Rr}{R+r} = \frac{r}{1 + \frac{r}{R}} = r$$

The equation of motion is thus

$$\frac{d^2\theta}{dt^2} = -\frac{20g}{13r^2}(r - \tfrac{5}{8}r)\theta = -\frac{15}{26}\cdot\frac{g}{r}\theta \quad . \quad . \quad . \quad . \quad (1)$$

Hence the time of a small oscillation is

$$\tau = 2\pi\sqrt{\frac{26r}{15g}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

EXAMPLE 2.—*A homogeneous solid sphere of radius* b *inside a fixed hollow sphere of radius* a.

Here $h = b$, $k^2 = \tfrac{2}{5}b^2$, $R = -a$, $r = b$. Hence the equation of motion is

$$\frac{d^2\theta}{dt^2} = -\frac{5g}{7b^2}\left(\frac{-ab}{b-a} - b\right)\theta$$

$$= -\frac{5}{7}\cdot\frac{g}{a-b}\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Therefore
$$\tau = 2\pi\sqrt{\frac{7(a-b)}{5g}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This sphere oscillates in the same time as a pendulum of length $\frac{7}{5}(a-b)$.

EXAMPLE 3.—*A plank of length* l *and of negligible thickness on a cylinder of radius* a. *The length of the plank is perpendicular to the cylinder.*

Here $h = 0$, $k^2 = \frac{1}{12}l^2$, $r = \infty$, $R = a$.
$$\frac{Rr}{R+r} = \frac{R}{\frac{R}{r}+1} = a$$

The equation of motion is therefore
$$\frac{d^2\theta}{dt^2} = -\frac{12g}{l^2}\cdot a\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Hence the time of oscillation is
$$\tau = 2\pi\sqrt{\frac{l^2}{12ag}} = \frac{\pi l}{\sqrt{3ag}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

EXAMPLE 4.—*A rigid body is formed by attaching two equal solid cylinders together by light rods clamped to their ends. The length and radius of each cylinder are* l *and* a *respectively. The axes are parallel and at a distance* 4a *apart. The rigid body thus formed is in equilibrium with the upper cylinder in contact with the highest point of a sphere of radius* a. *If a slight rolling disturbance perpendicular to the plane of the axes is given to the cylinders, to find the time of oscillation.*

In this case h is negative, because the centre of mass of the cylinders is below the point of support. Thus
$$h = -a, \; r = a, \; R = a.$$

We have yet to find the moment of inertia of the pair of cylinders about the generating line of the upper cylinder, which is in contact with the sphere in the equilibrium position.

Let $\frac{1}{2}M$ be the mass of each cylinder. The moment of inertia of the upper cylinder about the generator referred to is
$$\tfrac{1}{2}M\cdot(\tfrac{1}{2}a^2 + a^2) = \tfrac{3}{4}Ma^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The moment of inertia of the other cylinder about this same line, which is at a distance $3a$ from its axis, is
$$\tfrac{1}{2}M\{\tfrac{1}{2}a^2 + (3a)^2\} = \tfrac{19}{4}Ma^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Thus the whole moment of inertia is
$$\tfrac{3}{4}Ma^2 + \tfrac{19}{4}Ma^2 = \tfrac{11}{2}Ma^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The equation of motion for small oscillations is therefore
$$\tfrac{11}{2}Ma^2\frac{d^2\theta}{dt^2} = -Mg\left(\frac{a^2}{2a} + a\right)\theta$$

or
$$\frac{d^2\theta}{dt^2} = -\frac{3g}{11a}\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Hence the time for a small oscillation is

$$\tau = 2\pi\sqrt{\frac{11a}{3g}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

446. We will now work out one of two examples of oscillations of different types from the preceding ones.

EXAMPLE.—*A vessel resting on a horizontal table has a base which is part of a sphere of radius* a, *and the interior of the vessel is part of a concentric sphere of radius* b. *The mass of the vessel is* M, *and it contains a mass* m *of liquid. If the centre of mass of the vessel is at a distance* h *below the centre of the spheres, to find the time of a small rolling oscillation when the vessel is slightly disturbed, neglecting the friction between the liquid and the vessel.*

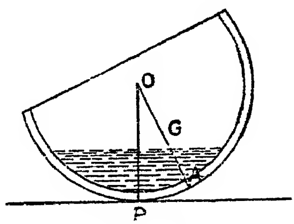


FIG. 194.

We shall deduce the equation of motion from the energy equation. This is often the easiest way of arriving at the equation of motion when dealing with a body composed of several parts which can move relatively to one another.

Let I′ be the moment of inertia of the body about a tangent line through A. Then since P is very near A, the moment of inertia of the body about the instantaneous axis through P differs very little from I′. Consequently, if θ is the angle through which the vessel has been displaced from the equilibrium position, the kinetic energy of the vessel alone is $\frac{1}{2}\text{I}'\left(\frac{d\theta}{dt}\right)^2$.

Now, since there is assumed to be no friction between the liquid and the vessel, the liquid cannot acquire any rotation. Neglecting the slight vertical motion due to the change of slope of the surface, its displacement at any instant is $a\theta$. Hence its kinetic energy is $\frac{1}{2}ma^2\left(\frac{d\theta}{dt}\right)^2$.

The work done on the system from the equilibrium position is just the work done by the weight of the vessel, namely, $-\text{M}gh(1-\cos\theta)$. Thus the energy equation is

$$\tfrac{1}{2}\text{I}'\left(\frac{d\theta}{dt}\right)^2 + \tfrac{1}{2}ma^2\left(\frac{d\theta}{dt}\right)^2 = -\text{M}gh(1-\cos\theta) + \text{C} \quad . \quad . \quad (1)$$

The constant C is the kinetic energy in the equilibrium position.

Differentiating both sides of (1) with respect to θ,

$$(\text{I}' + ma^2)\frac{d^2\theta}{dt^2} = -\text{M}gh\sin\theta$$

$$= -\text{M}gh\,\theta \text{ nearly for small oscillations} \quad (2)$$

Thus the period of a small oscillation is

$$\tau = 2\pi\sqrt{\frac{\text{I}' + ma^2}{\text{M}gh}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

427. *A body with a spherical base makes small oscillations on a perfectly smooth horizontal plane. The centre of mass is at a distance* h *below the centre of the sphere of which the base forms a part. To find the period of the oscillations.*

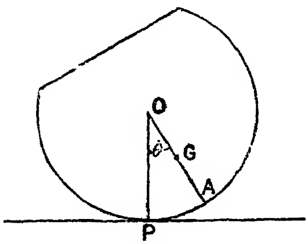


FIG. 195.

Since there is no friction there is no horizontal acceleration of the centre of mass G.

Assuming, then, that G had no horizontal velocity at the beginning of the motion, it will never have any afterwards. Thus G remains in the same vertical line throughout the motion.

The vertical distance of G below the horizontal plane through O is $h \cos \theta$. Hence the velocity of G downwards is

$$\frac{d}{dt}(h \cos \theta) = -h \sin \theta \cdot \frac{d\theta}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If k is the radius of gyration of the body about the axis through G perpendicular to the plane of motion, the kinetic energy of the body is

$$\tfrac{1}{2}\mathrm{M}k^2\left(\frac{d\theta}{dt}\right)^2 + \tfrac{1}{2}\mathrm{M}h^2 \sin^2 \theta\left(\frac{d\theta}{dt}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Since θ is always small, the second term in the kinetic energy, which contains the factor $\sin^2 \theta$, is very small compared with the first term. We may therefore neglect this term without much error. Then the energy equation is

$$\tfrac{1}{2}\mathrm{M}k^2\left(\frac{d\theta}{dt}\right)^2 = +\mathrm{M}gh \cos \theta + \mathrm{C} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Differentiating both sides with respect to θ,

$$\mathrm{M}k^2\frac{d^2\theta}{dt^2} = -\mathrm{M}gh \sin \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Therefore

$$\frac{d^2\theta}{dt^2} = -\frac{gh}{k^2}\theta \text{ nearly} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

This denotes an oscillation whose period is

$$\tau = 2\pi\sqrt{\frac{k^2}{gh}} = \frac{2\pi k}{\sqrt{gh}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The length of the equivalent simple pendulum is $\dfrac{k^2}{h}$.

For a solid hemisphere $h = \frac{3}{8}r$. Also the square of the radius of gyration about a diameter of the base is the same as for a sphere, namely, $\frac{2}{5}r^2$. Hence, since k is the radius of gyration about a parallel axis through the centre of mass,

$$k^2 = \tfrac{2}{5}r^2 - (\tfrac{3}{8}r)^2 = \tfrac{83}{320}r^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

CHAPTER XXII

IMPULSES AND INITIAL MOTIONS

448. WHEN a large force acts on a body for a very short time the acceleration produced is very large, and consequently there may be an appreciable alteration in velocity during the short time that the force acts. Such forces occur when two bodies collide. For example, when a marble falls on a stone floor the marble seems to reverse its direction of motion suddenly, that is, in an infinitely short interval of time. But the interval of time during which the force acts, although short, is not infinitely short, for that would require the force to be infinitely large. Yet the time is so short that, for most calculations, it may be considered zero. Theoretically, for collisions, we consider the average force to be infinitely large and the time of action infinitely short, while their product is finite. This product of force and time is called an *impulse*. If the interval is not so short that we may consider it to be infinitely short, the impulse is the integral of the force with respect to time.

Suppose a particle of mass m is moving in a straight line with a velocity u, and a force F acts on it in the direction of motion. Then

$$m\frac{du}{dt} = \mathrm{F} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Hence

$$mu - mu_0 = \int_0^\tau \mathrm{F}\,dt. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

u and u_0 being the velocities when t was 0 and τ respectively.

The quantity

$$\int_0^\tau \mathrm{F}\,dt$$

is what we have called the impulse of the force F. Actually, the name impulse is not used except when the force is large and τ small. Then the force F is called an *impulsive force*. If we write P for the impulse, equation (2) may be written

$$mu - mu_0 = \mathrm{P} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

that is,

$$\text{increase of momentum} = \text{impulse} \quad . \quad . \quad . \quad . \quad (4)$$

Impulses have direction and magnitude just as forces have, and they are added by vector rules.

One very important principle to remember is that, in dealing with the effect produced by impulses, we may neglect entirely the effect of finite forces during the short interval in which the impulses act.

449. Impulses acting on a Rigid Body.—Suppose forces F_1, F_2, F_3, etc., parallel to one plane, act on a rigid body of mass M; and let p_1, p_2, p_3, be the perpendiculars from the centre of gravity of the body on the lines of action of these forces. Let u and v denote the component velocities of the centre of gravity parallel to a pair of rectangular axes OX and OY in the plane of its motion. Let ϕ_1, ϕ_2, ϕ_3, etc., be the angles which F_1, F_2, F_3, etc., make with OX. Then

$$M\frac{du}{dt} = F_1 \cos \phi_1 + F_2 \cos \phi_2 + F_3 \cos \phi_3 + \text{etc.}$$

$$= \Sigma F \cos \phi, \text{ say} \quad \ldots \ldots \ldots \quad (1)$$

$$M\frac{dv}{dt} = \Sigma F \sin \phi \quad \ldots \ldots \ldots \quad (2)$$

$$I\frac{d\omega}{dt} = \Sigma pF \quad \ldots \ldots \ldots \quad (3)$$

Now, let us suppose that the forces F_1, F_2, F_3, etc., are very large, and the interval during which they act is very small. Denoting this interval by τ and the impulse of F_1 by P_1, we get

$$\int_0^\tau F_1 \cos \phi_1 \, dt = \cos \phi_1 \int_0^\tau F_1 dt = P_1 \cos \phi_1 \quad \ldots \quad (4)$$

Also
$$\int_0^\tau p_1 F_1 dt = p_1 \int_0^\tau F_1 dt = p_1 P_1 \quad \ldots \ldots \quad (5)$$

Even if the force F_1 is applied at a fixed point in the rigid body (in which case p_1 will vary), the interval is supposed to be so short that p_1 has no time to alter appreciably. For this reason p_1 is regarded as constant in (5).

Now, integrating (1), (2), and (3), with respect to t, we find

$$Mu - Mu_0 = \Sigma P \cos \phi \quad \ldots \ldots \quad (6)$$

$$Mv - Mv_0 = \Sigma P \sin \phi \quad \ldots \ldots \quad (7)$$

$$I\omega - I\omega_0 = \Sigma pP \quad \ldots \ldots \quad (8)$$

The equations express the fact that the increase in the linear momentum of the body, that is, in the momentum of a particle of mass M moving with the centre of gravity, is equal to the resultant of the impulses. The last equation expresses the fact that the increase in moment of momentum about the centre of mass is equal to the sum of the moments of the impulses.

Since an impulse is the time-integral of a force, it is obvious that an impulse equation can be derived from any form of equation of motion by merely integrating with respect to time. That is, in dealing with impulses we may do all that can be done with forces, provided that we replace rate of change of momentum in an equation of motion by actual change of momentum.

450. *A free uniform rod of length* 2a *is struck by a blow whose impulse is* P *acting in a line perpendicular to the rod at a distance* c *from its mid-point. To find the motion just after the blow, assuming it was at rest just before the blow.*

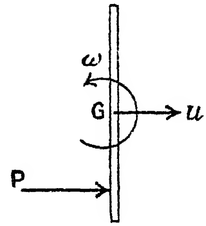


FIG. 196.

If u and ω are the velocity of the centre of mass and the angular velocity just after the blow,

$$Mu = P \quad . \quad . \quad . \quad . \quad (1)$$

$$M \, . \, \tfrac{1}{3}a^2\omega = cP \quad . \quad . \quad . \quad . \quad (2)$$

These equations give both u and ω when P is given. But if the magnitude of P is not given, the equations give the relation between u and ω. Thus

$$u = \frac{P}{M} = \frac{a^2}{3c}\omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which gives u in terms of ω, or *vice versâ*.

Suppose the blow is struck at the end of the rod, so that $c = a$, then

$$u = \tfrac{1}{3}a\omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

451. *A body of any form is attached to a fixed axis through* A, *and it receives a blow* P *in a direction perpendicular to the axis and making an angle* ϕ *with the plane containing the axis and the centre of mass* G. *To find the angular velocity after the impulse and the action at the hinge.*

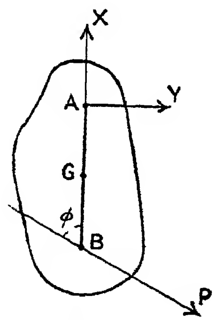


FIG. 197.

Let AG $= h$, GB $= c$, $k =$ the radius of gyration about the axis through G parallel to the fixed axis.

Let ω be the angular velocity just after the blow, and ω_0 before the blow. The velocity of the centre of mass just after the blow is $h\omega$.

X and Y are the components of the impulse which the axis exerts on the body in consequence of the blow P.

For the motion of the centre of mass G perpendicular to and along AG,

$$Mh(\omega - \omega_0) = P \sin\phi + Y \quad . \quad (1)$$

$$0 = P\cos\phi - X \quad . \quad (2)$$

The left-hand side of (2) is zero because the velocity of G along AG is zero.

For the rotation, we get the equation

$$Mk^2(\omega - \omega_0) = cP\sin\phi - hY \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Equation (2) gives X in terms of P. Also, eliminating Y from (1) and (3), we get

$$M(k^2 + h^2)(\omega - \omega_0) = (c + h)P\sin\phi \quad . \quad . \quad . \quad (4)$$

whence

$$\omega - \omega_0 = \frac{P}{M} \cdot \frac{c + h}{k^2 + h^2}\sin\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Substituting for $(\omega - \omega_0)$ in either (1) or (3), we find

$$Y = \frac{ch - k^2}{k^2 + h^2} \cdot P \sin \phi \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

If $P \sin \phi$ is not zero, Y will be zero only when

$$ch = k^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

that is, when the point B is the centre of oscillation for the axis A. If the resultant impulse P were perpendicular to AB, and B were the centre of oscillation, there would be no reaction between the axis and the body. For this reason the centre of oscillation is sometimes called the *Centre of Percussion* corresponding to the axis A.

Another method of arriving at the angular velocity very easily is by taking moments about the axis.

The moment of the impulse about this axis is equal to the actual change of moment of momentum. That is

$$M(k^2 + h^2)(\omega - \omega_0) = P(AB \sin \phi) = P(h + c) \sin \phi \quad . \quad (8)$$

which is just the same equation as (4).

As a particular example, suppose a uniform rod turns about a horizontal axis at one end perpendicular to the rod. The rod is allowed to fall from a position in which the free end is higher than the axis. When it becomes horizontal, a point very near the free end strikes a fixed bar. We will compare the impulses exerted by the bar and the axis.

Let P be the impulse exerted by the fixed bar, and Y that exerted by the axis as in the earlier part of this article. X is clearly zero. Let l denote the length of the rod.

Here $\phi = 90°$, $k^2 = \frac{1}{12}l^2$, $h = \frac{1}{2}l$, $c = \frac{1}{2}l$. Hence

$$Y = \frac{\frac{1}{4}l^2 - \frac{1}{12}l^2}{\frac{1}{12}l^2 + \frac{1}{4}l^2} P = \frac{1}{2}P \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

This result is just the same, whether the rod rebounds or not. In this example ω_0 is negative, and ω is either positive or zero according as the rod rebounds or not.

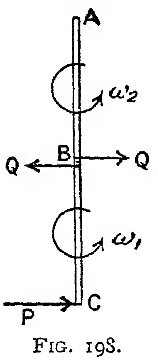


FIG. 198.

452. *Two equal uniform rods,* AB, BC, *each of length* 2a *and mass* m, *are hinged together at* B, *and are in one straight line at rest. An impulse* P *is applied at* C *perpendicular to the line of the rods. To find the motion immediately after the blow.*

Here we shall have to deal with the motion of each rod separately, taking into account the reaction between the rods at the hinge. We shall also have one geometrical relation between the motions of the rods, namely, that due to the fact that the velocity of B is the same when considered as a part of either rod.

Q is the reaction between the rods as indicated in the figure. Let u_1 and u_2 be the velocities, just after the blow, of the centres of mass of BC and AB in the direction of P; let ω_1 and ω_2 be their angular velocities in the direction indicated in the figure.

The velocity of B considered as a part of AB is $u_2 + a\omega_2$, and considered as a part of BC it is $u_1 - a\omega_1$. Hence

$$u_2 + a\omega_2 = u_1 - a\omega_1 \quad \ldots \ldots \quad (1)$$

Now for the motion of AB

$$mu_2 = Q \quad \ldots \ldots \ldots \quad (2)$$

$$\tfrac{1}{3}ma^2\omega_2 = aQ \quad \ldots \ldots \ldots \quad (3)$$

And for the motion of BC

$$mu_1 = P - Q \quad \ldots \ldots \ldots \quad (4)$$

$$\tfrac{1}{3}ma^2\omega_1 = aP + aQ \quad \ldots \ldots \ldots \quad (5)$$

Immediately after the blow there is obviously no velocity parallel to the line of the rods, and therefore no other equations are needed.

Now substituting in (1) the velocities given by the equations (2), (3), (4), (5), we get, on multiplying by $3m$,

$$Q + 3Q = (P - Q) - 3(P + Q) \quad \ldots \ldots \quad (6)$$

whence

$$Q = -\tfrac{1}{4}P \quad \ldots \ldots \ldots \ldots \quad (7)$$

Thus Q acts in the direction opposite to that assumed in the figure.

On substituting for Q in the equations (2) to (5) we find

$$u_2 = -\frac{1}{4} \cdot \frac{P}{m} \quad \ldots \ldots \ldots \quad (8)$$

$$\omega_2 = -\frac{3}{4a} \cdot \frac{P}{m} \quad \ldots \ldots \ldots \quad (9)$$

$$u_1 = \frac{5}{4} \cdot \frac{P}{m} \quad \ldots \ldots \ldots \quad (10)$$

$$\omega_1 = \frac{9}{4a} \cdot \frac{P}{m} \quad \ldots \ldots \ldots \quad (11)$$

It follows from these results that the two rods begin to rotate in opposite directions, and their centres of mass begin to move in opposite directions.

453. *One end of a uniform rod comes in contact with a smooth horizontal floor while the centre of gravity is moving vertically downwards. Just before striking the floor the rod is rotating, and is inclined at θ to the floor. To find the change of motion due to the impulse, assuming that the end which strikes the floor does not rise again.*

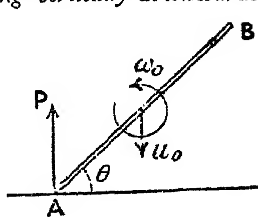


FIG. 199.

Let u_0 be the velocity of the centre of gravity and ω_0 the angular velocity of the rod just before striking the floor. Let u and ω be the values of these quantities just after the impulse, the signs being positive if the directions are the same as those of u_0 and ω_0. Let P denote the impulse, $2a$ the length of the rod.

Resolving vertically we get

$$m(u - u_0) = -P \quad \ldots \ldots \ldots \quad (1)$$

Taking moments about the centre of mass

$$\tfrac{1}{3}ma^2(\omega - \omega_0) = - Pa\cos\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Since the end A does not rise off the floor the vertical component of the velocity of A just after the impulse is zero. Now the velocity of A relative to the centre of mass is $a\omega$ perpendicular to BA, and the vertical (downward) component of this is $a\omega\cos\theta$. Hence

$$u + a\omega\cos\theta = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Substituting in (3) the values of u and ω taken from (1) and (2)

$$u_0 - \frac{P}{m} + a\omega_0\cos\theta - 3\frac{P}{m}\cos^2\theta = 0 \quad . \quad . \quad . \quad . \quad (4)$$

Whence
$$\frac{P}{m} = \frac{(u_0 + a\omega_0\cos\theta)}{1 + 3\cos^2\theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Now (1) gives
$$u = \frac{(3u_0\cos\theta - a\omega_0)\cos\theta}{1 + 3\cos^2\theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and (3) gives
$$\omega = \frac{1}{a}\cdot\frac{a\omega_0 - 3u_0\cos\theta}{1 + 3\cos^2\theta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Very simple reasoning tells us that, if $\theta = 90°$, u will be zero and ω will remain unchanged since the floor is smooth. Equations (6) and (7) give these expected results.

454. *A free rigid body has a motion of translation and a rotation about an axis through the centre of mass perpendicular to the motion of translation. An axis parallel to the axis of rotation is suddenly brought to rest. To find the motion immediately after the fixing of the axis, and the impulse exerted by the axis.*

Let ω_0 be the angular velocity of the body and u_0 the velocity of the centre of mass just before fixing the axis. Let axes GX and GY be taken parallel and perpendicular to u_0, and let the co-ordinates of the axis which becomes fixed be a and b. P and Q are the components of the impulse exerted by the axis on the body. M is the mass of the body, Mk^2 the moment of inertia about the axis of rotation through G.

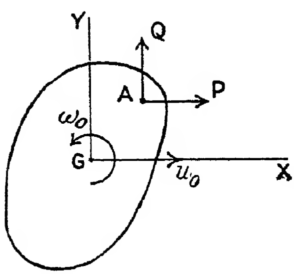


FIG. 200.

When the axis A becomes fixed the body rotates about A, and therefore the velocity of G will be AG . ω, where ω is the angular velocity after the impulse.

While the impulsive force is acting at A the rate of change of moment of momentum of the body about A is zero. Even if some finite force does act on the body during this interval its effect may be neglected, as we pointed out in Art. 448. Consequently, the whole change of moment of momentum about A is zero while the impulse acts.

By Art. 405 the moment of momentum about A before the impulse is

$$bMu_0 + Mk^2\omega_0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and after the impulse

$$AG \, . \, M \, . \, AG\omega + Mk^2\omega = M(a^2 + b^2)\omega + Mk^2\omega \quad . \quad . \quad (2)$$

Since there is no change of moment of momentum about A,

$$M(bu_0 + k^2\omega_0) = M(a^2 + b^2 + k^2)\omega \quad . \quad . \quad . \quad . \quad (3)$$

whence

$$\omega = \frac{bu_0 + k^2\omega_0}{a^2 + b^2 + k^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

To find P and Q we resolve horizontally and vertically for the motion of G. The components of the velocity of G parallel to GX and GY are, after the impulse, $b\omega$ and $-a\omega$. Hence

$$P = M(b\omega - u_0) = \frac{bk^2\omega_0 - (k^2 + a^2)u_0}{a^2 + b^2 + k^2}M \quad . \quad . \quad . \quad (5)$$

$$Q = -Ma\omega = -\frac{a(bu_0 + k^2\omega_0)}{a^2 + b^2 + k^2}M \quad . \quad . \quad . \quad . \quad . \quad (6)$$

We have deduced (3) from the constancy of the moment of momentum about A. Although this gives the simplest solution of the problem, it is not necessary to assume it. We could have obtained (3) from the three equations for change of motion, namely (5), (6), and the equation (7) below, obtained by taking moments about G.

$$Mk^2(\omega - \omega_0) = aQ - bP \quad . \quad . \quad . \quad . \quad . \quad (7)$$

If we substitute in (7) the first expressions for P and Q from (5) and (6) we shall get equation (3).

455. *Two equal uniform rods* AB, BC, *hinged together at* B, *fall vertically as one straight rod* AC *without rotation and inclined at an angle* α *with the vertical. The lower end comes in contact with a rough horizontal floor when the common velocity of the rods is* V. *Assuming that* A *remains at rest just after striking the floor, and that the hinge is perpendicular to the vertical plane containing the rods, to find the motion after the impulse.*

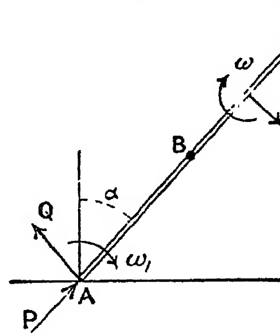


FIG. 201.

Let $2a$ be the length of each rod, and m the mass of each. Let the velocity of the centre of mass of BC just after the impulse be u. This velocity is clearly perpendicular to BC. Let ω_1 and ω denote the angular velocities of AB and BC just after the impulse. The velocity of the centre of mass of AB is $a\omega_1$ and it is perpendicular to AB.

Now the change in the moment of momentum of each particle of the two rods about A is equal to the moment about A of the impulse on

the particle. Consequently the total change in the moment of momentum of the two rods about A is equal to the sum of the moments about A of all the impulses on all the particles, which reduces to the moment of the external impulses, since the reactions between the particles will disappear from the sum. But the only external impulse is the reaction of the floor and this has no moment about A. Consequently there is no change in the moment of momentum of the two rods about A.

Before the impulse the moment of momentum of the two rods about A was clearly $2m \,.\, 2aV \sin \alpha$. After the impulse the moment of momentum of AB about A is $ma^2\omega_1 + \frac{1}{3}ma^2\omega_1$, or $\frac{4}{3}ma^2\omega_1$; and the moment of momentum of the rod BC is $3a \,.\, mu + \frac{1}{3}ma^2\omega$. Hence, equating the sum of the moments of momentum before and after the impulse,

$$4maV \sin \alpha = \tfrac{4}{3}ma^2\omega_1 + 3mau + \tfrac{1}{3}ma^2\omega \quad . \quad . \quad . \quad (1)$$

or

$$12V \sin \alpha = 4a\omega_1 + 9u + a\omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Now also since the only impulse on the rod BC acts at B the moment of momentum of this rod about B remains unchanged. That is,

$$maV \sin \alpha = mau + \tfrac{1}{3}ma^2\omega \quad . \quad . \quad . \quad . \quad . \quad (3)$$

or

$$3V \sin \alpha = 3u + a\omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

We have three unknown velocities to find, namely, ω_1, ω, and u. At present we have two equations, (2) and (4), connecting these velocities. We need still another equation. We get this other equation from the geometrical condition that B is attached to each rod, and therefore has the same velocity when considered as a part of either rod. This condition gives

$$2a\omega_1 = u - a\omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Substituting for $a\omega_1$ in (2) we get

$$\begin{aligned} 12V \sin \alpha &= 2(u - a\omega) + 9u + a\omega \\ &= 11u - a\omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6) \end{aligned}$$

Solving (4) and (6), we get

$$u = \tfrac{15}{14}V \sin \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$a\omega = -\tfrac{3}{14}V \sin \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Then (5) gives

$$a\omega_1 = \tfrac{9}{14}V \sin \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

If we now want to find the reaction of the floor we can find the impulse which will produce the known change of momentum. Let P and Q be the components, along and perpendicular to AB, of the impulse at A. These are equal to the total increase in the momentum of the two bodies in the same two directions. Thus

$$P = 2mV \cos \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$\begin{aligned} Q &= -(mu + ma\omega_1) + 2mV \sin \alpha \\ &= \tfrac{2}{7}mV \sin \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11) \end{aligned}$$

456. *A lift is raised and lowered by means of four ropes attached to the top corners which lie at the angular points of a square. If the mid-point of one of the bottom edges is suddenly brought to rest by meeting an obstacle just after the lift had been moving downwards with velocity* v_0, *to find the impulse of the tension in the two ropes which do not become slack.*

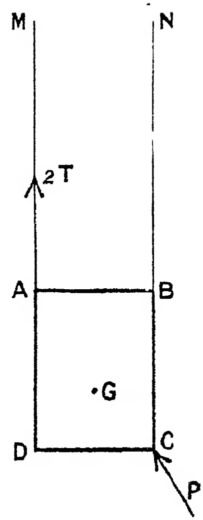


FIG. 202.

The figure represents an elevation perpendicular to the edge which becomes fixed. This edge is represented by C in the figure, and the two strings which do not become slack are represented by the straight line MA.

The answer to this question depends on the assumption we make concerning the behaviour of the point A. We shall assume that the ropes are inextensible, and that the body which gives out the ropes does not yield in the impulse. Then, if the ropes are attached directly to the lift without the intervention of springs, we must assume that the downward velocity of A remains unchanged, because the speed at which the rope is being given out does not alter during the impulse.

Let CD = a, CG = h; and let k denote the radius of gyration about the axis through G perpendicular to the plane of the figure. Let ω be the angular velocity after impact.

The velocity of A after impact is AC . ω perpendicular to AC, since C is fixed. The downward component of this is DC . ω. But the downward component of the velocity of A is v_0. Hence

$$a\omega = v_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now the change in the moment of momentum about C is equal to the moment of 2T about C. Before impact the moment of momentum was $\frac{a}{2}Mv_0$, and after impact it is $hMh\omega + Mk^2\omega$. Hence

$$M(h^2 + k^2)\omega - \frac{a}{2}Mv_0 = -2aT \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From (1) and (2), on eliminating ω,

$$Mv_0\left(\frac{h^2 + k^2}{a} - \frac{a}{2}\right) = -2aT \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

or

$$T = \frac{Mv_0}{2}\left(\frac{1}{2} - \frac{h^2 + k^2}{a^2}\right) \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If this expression for T happens to be negative, it would mean that the motion we have supposed would require a thrust in the ropes MA. Since this is not possible, these ropes, as well as the ropes NB, will become slack.

If the centre of mass G does not lie in the vertical plane containing

the impulse P, then the sum of the tensions in the two ropes MA will be what we have called $2T$ above, but these two tensions will not be equal.

457. If the ropes in the last example were elastic there would be no real impulse in the ropes MA. The ropes would stretch and the tension would quickly increase up to a maximum and then it would decrease again, and oscillations would be set up which would soon die out through friction. This illustrates what happens in every case of impulse or collision between two bodies. If a blow be suddenly applied to one point of an elastic body, the particles at which the blow is applied will yield to the blow and these will push forward the next particles, and these again will push the next, and so on till the motion communicated by the blow has spread over the whole body. At the first instant of the blow it is only the particles in immediate contact that are in motion, and it takes some measurable, although very short, time for the effect of the blow to be transmitted through the body. All apparently rigid bodies are set in motion by an impulse in this way, and even with such plastic things as clay the process is similar, but it is accompanied by a permanent change of shape.

When two elastic bodies collide they always rebound after collision. The reason for the rebound may be explained as follows. The collision causes a compression of the particles of each body near where the blow is applied, and, as explained above, these particles push forward the next particles, and these the next, and so on. When the compression-wave which sets the particles in motion arrives at a boundary of the body, it leaves these last particles with some velocity outwards from the boundary, which is greater than the general velocity of the whole body. The particles travel on until the body is extended at that point, just as if there had been applied a sudden pull at that point. Just as the compression-wave travelled forward through the body so the extension-wave will travel backwards, and when it arrives at the point at which the blow was applied it will draw itself away from the other body if it is still in contact. The separation may have taken place already on account of the extension-wave from the other body.

458. Collision of Elastic Bodies.—When two elastic spheres collide their component velocity of separation after impact parallel to that line which passes through their centres when they are in contact bears to the component velocity of approach before impact parallel to that line a ratio which depends only on the materials of which the spheres are made. This ratio is called the *coefficient of restitution* or *coefficient of resilience* for the two spheres. The value of this coefficient may have any magnitude between nought and unity. It is very nearly zero if one of the spheres is made of soft clay. For two lead spheres it is about 0·2, and for two glass spheres about 0·94.

When two elastic bodies of any form collide, the points which come in contact rebound after the impact. There is a similar definition for the coefficient of restitution in this case, but now the velocities, whose ratio is taken, are the component velocity of separation of the points of contact along the common normal to the surfaces of contact and the

component velocity of approach of these points along that same line. But in the general case the coefficient of restitution depends on the particular points which come in contact, the form of the colliding bodies, and the position of the bodies relative to the normal at the surface of contact as well as the material of the bodies. On account of the large number of variable quantities in the coefficient of restitution of two elastic bodies these coefficients have only been determined for particular cases, such as the impact of two spheres, the impact of a sphere on a plane surface of a large body, and the impact of two rods in the same line moving parallel to their lengths and meeting at their ends.

The coefficient of restitution will be denoted by e. It can be shown in the general case that there is always kinetic energy lost on impact if e is less than unity. When e is equal to unity the total kinetic energy after impact is equal to the total kinetic energy before impact, but since e is less than unity for all actual cases there is always some loss of kinetic energy.

In the next article we will show, in a fairly simple case, that there is no change in the kinetic energy when e is unity.

459. *A rod which has a motion of translation and rotation in the same plane is struck by a ball whose centre of mass is also moving in the same plane. To find the subsequent motion, the impulse, and to show that there is no change of kinetic energy, assuming that* e = 1, *and that the bodies are smooth.*

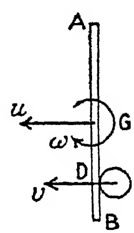


FIG. 203.

The angular velocity of the rod after impact is ω, and before impact ω_0 in the same direction. The component velocities of the rod and the ball in the direction of the impulse on the rod are u, v, after impact, and u_0, v_0, before impact. The component velocities perpendicular to the impulse remain unaltered.

Let P denote the impulse on the rod, and D the point at which it is applied. The impulse on the ball is P in the opposite direction. Let M and m be the masses of the rod and ball. Let GD $= c$.

The equations for the change of motion of the rod are

$$M(u - u_0) = P \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$Mk^2(\omega - \omega_0) = cP \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

And for the ball

$$m(v - v_0) = -P \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The velocity of approach of the point D of the rod and ball before impact, resolved along the perpendicular to the surface of contact, that is, perpendicular to the rod, is $v_0 - (u_0 + c\omega_0)$. The velocity of separation after impact is a similar expression with the signs changed and the suffixes dropped. Hence, since $e = 1$, we get

$$v_0 - (u_0 + c\omega_0) = (u + c\omega) - v \quad . \quad (4)$$

or $$(u + u_0) + c(\omega + \omega_0) - (v + v_0) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Therefore

$$P(u + u_0) + cP(\omega + \omega_0) - P(v + v_0) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Now let the three values of P obtained from the equations (1), (2), (3), be substituted for the first, second, and third P in (6). Then

$$M(u^2 - u_0^2) + Mk^2(\omega^2 - \omega_0^2) + m(v^2 - v_0^2) = 0 \quad . \quad (7)$$

That is

$$\tfrac{1}{2}(Mu^2 + Mk^2\omega^2 + mv^2) = \tfrac{1}{2}(Mu_0^2 + Mk^2\omega_0^2 + mv_0^2) \quad (8)$$

Thus the kinetic energy of the motion parallel to the impulse remains unaltered after the impulse, and since the velocities perpendicular to the impulse remain unchanged, it follows that the total kinetic energy of the two bodies is unaltered by the impulse.

Now to find P.

Substituting in (5) the values of u, v, and ω, from (1), (2), and (3), we get

$$\frac{P}{M} + 2u_0 + \frac{c^2}{k^2}\cdot\frac{P}{M} + 2c\omega_0 + \frac{P}{m} - 2v_0 = 0 \quad . \quad . \quad . \quad (9)$$

Therefore $$P\left\{M + m\left(1 + \frac{c^2}{k^2}\right)\right\} = 2Mm(v_0 - u_0 - c\omega_0) \quad . \quad . \quad (10)$$

This gives P, and on putting its value in (1), (2), and (3), the velocities u, v, and ω, can be found immediately.

If e is less than 1 it is easy to show that there is a loss of kinetic energy in the two bodies. Instead of (4) we should have

$$e\{v_0 - (u_0 + c\omega_0)\} = u + c\omega - v \quad . \quad . \quad . \quad . \quad (11)$$

whence

$$(u + u_0) + c(\omega + \omega_0) - (v + v_0) = -(1 - e)\{v_0 - (u_0 + c\omega_0)\} \quad (12)$$

The left-hand side of this has to be treated exactly as in deducing (6) and (7) from (5). Then we get

$$\tfrac{1}{2}(Mu^2 + Mk^2\omega^2 + mv^2) - \tfrac{1}{2}(Mu_0^2 + Mk^2\omega_0^2 + mv_0^2)$$
$$= -\tfrac{1}{2}(1 - e)\{v_0 - (u_0 + c\omega)\}P \quad . \quad . \quad . \quad . \quad (13)$$

The right-hand side of this equation must be negative, because the quantity in the large brackets is the velocity of approach before the impact, and P and $(1 - e)$ are both positive. Hence the left-hand side, which is the gain of kinetic energy, is negative.

460. *In a game of cricket the bowler delivers the ball in a horizontal direction at a height* 6 *feet above the ground. When the ball is delivered it has a spin about the axis perpendicular to the plane of motion of its centre of mass, the direction of the spin being such that the friction retards the motion when the ball strikes the ground. The point at which the ball strikes the ground is* 18 *yards from the bowler and* 4 *yards from the batsman's wickets. If the coefficient of friction between the ball and the ground is* $\frac{1}{3}$, *and the coefficient of restitution* $\frac{1}{2}$, *at what height could the ball hit the wickets if it were allowed to go?*

Let u and v be the horizontal and vertical components of the velocity when the ball strikes the ground. Then

$$v = \sqrt{2g \cdot 6} = 8\sqrt{6} \text{ feet per sec.} \quad . \quad . \quad . \quad (1)$$

The equation of the parabolic path with the origin at the highest point is

$$y = \frac{g}{2u^2}x^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence $$u = \sqrt{\frac{gx^2}{2y}} = \sqrt{\frac{32 \times 54^2}{12}} = 36\sqrt{6} \quad . \quad . \quad . \quad (3)$$

Let v_1 and u_1 be the vertical and horizontal velocities after the impact. Then, if P is the impulse,

$$P = m(v + v_1) = m(v + ev) = m \cdot \tfrac{3}{2} \cdot 8\sqrt{6} = 12\sqrt{6}m \quad . \quad (4)$$

If the rotation is not annihilated by the friction, then there is limiting friction acting during the whole of the time of contact, and the impulse of this friction is μP. Hence

$$m(u_1 - u) = -\mu P = -\tfrac{1}{3}P = -\tfrac{1}{3} \times 12\sqrt{6}m = -4\sqrt{6}m \quad (5)$$

Therefore $$u_1 = u - 4\sqrt{6} = 32\sqrt{6} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

If t is the number of seconds between the instants when the ball leaves the ground and strikes the wickets, that is, the time taken to travel 12 feet with a velocity u_1

$$t = \frac{12}{u_1} = \frac{12}{32\sqrt{6}} = \frac{\sqrt{6}}{16} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

If h is the height at which the ball strikes the wickets

$$h = v_1 t - \tfrac{1}{2}gt^2 = (4\sqrt{6})\frac{\sqrt{6}}{16} - \frac{6}{16} = \frac{9}{8}\text{ feet} \quad . \quad . \quad . \quad (8)$$

461. Initial Motions.—If one of the supports of a body, which is at rest or moving in any manner, suddenly gives way or is removed, the reactions at the other supports are instantaneously altered. The reason for this alteration is that the accelerations of the body are changed when the support is removed. In order to find these altered reactions we have only to consider the motion of the body immediately after the removal of the support; and since it is only the reactions at the beginning of the new motion that are required, we can simplify our equations by assuming that all the displacements and velocities due to the new accelerations are infinitely small while these accelerations are finite. As a simple example we may mention the case of a particle allowed to fall from rest under gravity. This does not illustrate change in reactions, but it is an illustration of the fact that displacement and velocity may be zero while acceleration is finite.

The sudden removal of a support produces just the same effect as the sudden application of a force which is exactly the reverse of that exerted by the support while all the other forces remain the same. But it must not be imagined from this that a suddenly applied force is in any way comparable with an impulse. An ordinary finite force will require a finite time to generate a finite velocity, whereas an impulse is an action which produces finite changes of velocity instantaneously.

No new principles are needed to deal with questions on initial motions. We shall, however, always have a geometrical equation connecting the displacements, and since it is only the initial motion that is being investigated, we shall be able to neglect everything except finite quantities and the first powers of the displacements, since the second powers and products will be infinitely small in comparison with the first powers.

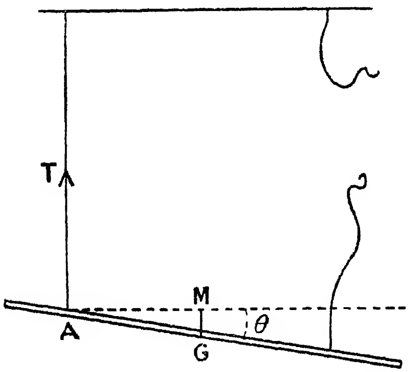


FIG. 204.

462. *A uniform horizontal bar is supported by two vertical strings attached to points at distance* c *from the mid-point of the rod. If one of the strings suddenly breaks, to find the tension in the other string immediately after breaking and to compare it with the tension before breaking.*

T is the tension after the breaking of one of the strings, T_0 the tension before.

Let y be the downward displacement of the centre of gravity in a very short interval of time after the breaking, and let θ be the angle through which the rod has turned. Also let $2a$ denote the length of the rod. Then, for the motion of G,

$$M\frac{d^2y}{dt^2} = Mg - T \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and for the rotation
$$\tfrac{1}{3}Ma^2\frac{d^2\theta}{dt^2} = cT \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Now the geometrical equation connecting y and θ is obviously

$$y = c\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

since it has only to be correct for the first powers of y and θ Differentiating (3) twice, we get

$$\frac{d^2y}{dt^2} = c\frac{d^2\theta}{dt^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

On eliminating the accelerations from (1), (2), and (4), we get

$$\tfrac{1}{3}a^2(Mg - T) = c^2T \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

whence
$$T = \frac{\frac{1}{3}a^2}{\frac{1}{3}a^2 + c^2}Mg = \frac{a^2}{a^2 + 3c^2}W \quad . \quad . \quad . \quad (6)$$

where W is the weight of the rod.

For equilibrium before breaking

$$T_0 = \tfrac{1}{2}W \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Hence
$$\frac{T}{T_0} = \frac{2a^2}{a^2 + 3c^2} = 1 + \frac{a^2 - 3c^2}{a^2 + 3c^2} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Thus T is greater or less than T_0 according as a^2 is greater or less than $3c^2$.

If the strings are attached at the ends of the rod, then $c = a$, and

$$T = \tfrac{1}{4}W = \tfrac{1}{2}T_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

463. *A rigid body of weight* W *is suspended by two vertical strings attached at distances* a *and* b *from the vertical through the centre of gravity. To find the tension in one string immediately after the other breaks.*

Suppose the string at distance b breaks. Let k be the radius of gyration of the body about the axis through G perpendicular to the plane of motion.

Since all the forces are initially vertical the centre of gravity begins to move vertically downwards. Also since the point of attachment of the unbroken string cannot move vertically, it follows that the instantaneous centre of rotation at the beginning of the motion is in the line of this string produced. To make the centre of gravity move vertically this instantaneous centre must be in the horizontal plane through the centre of gravity. Hence, if y is the downward displacement of the centre of gravity, and θ the angular displacement after a very short interval of time, it is clear that, to the first order of small quantities

$$y = a\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now the equations of motion are

$$M\frac{d^2y}{dt^2} = Mg - T \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$Mk^2\frac{d^2\theta}{dt^2} = aT \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Substituting in (2) the value of y from (1), and then eliminating $\frac{d^2\theta}{dt^2}$ from the resulting equation and (3),

$$k^2(Mg - T) = a^2T$$

or

$$T = \frac{k^2}{k^2 + a^2}Mg = \frac{k^2}{k^2 + a^2}W \quad . \quad . \quad . \quad (4)$$

If T_0 was the tension in this string before the other broke

$$T_0 = \frac{b}{a + b}W \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Hence

$$\frac{T}{T_0} = \frac{k^2}{k^2 + a^2} \cdot \frac{a + b}{b} = 1 + \frac{a(k^2 - ab)}{(k^2 + a^2)b} \quad . \quad . \quad . \quad (6)$$

Thus, T is greater or less than T_0 according as k^2 is greater or less than ab. If k^2 is equal to ab, then the point in which the horizontal plane through G meets the line of the broken string just before the motion began is the centre of percussion corresponding to the point in which the line of the other string is met by the same horizontal plane through G.

464. Body supported by Three Strings or Props, one of which is removed.—If a body is supported by three vertical strings and one of them suddenly breaks or is removed, the problem of finding the initial tension in the other two can be reduced to the problem when one of two strings breaks. For, obviously, both for rest and initial motion the two unbroken strings can be replaced by a single equivalent string in the same vertical plane as the third string and the centre of mass of the body. We have then to find the tension in this equivalent string, and to resolve this tension into two equivalent forces along the two given strings.

Suppose a circular disc of radius r is suspended by three vertical strings attached at equidistant points on the circumference. When one of the strings breaks, to find the tensions in the other two.

We can use the result of the last article. If the thickness of the disc is negligible $k^2 = \frac{1}{4}r^2$. Also for the equivalent string $a = \frac{1}{2}r$; and b, the distance of the broken string from the centre of gravity, is r. Then by the last article

$$T = \frac{\frac{1}{4}r^2}{\frac{1}{4}r^2 + \frac{1}{4}r^2}W = \frac{1}{2}W \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

But T is the tension in the equivalent string which is twice the tension in each actual string. Thus the tension T_1 in each unbroken string is $\frac{1}{4}$W. Before the breaking of the string the tension in each was $\frac{1}{3}$W. It is seen then that the tensions in the unbroken strings are diminished to three-quarters of their equilibrium value.

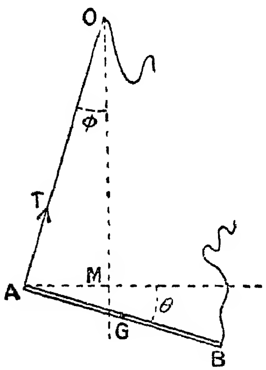


FIG. 205.

465. *A uniform rod* AB *of length* 2a *is suspended by two strings* OA, OB, *each of length* l. *If one of them breaks, to find the initial tension in the other.*

Here the centre of mass has an initial horizontal as well as vertical acceleration, because the tension of the unbroken string has a horizontal component.

Let ϕ be the angle which the unbroken string makes with the vertical after a very short interval of time t from the beginning of the motion, and let θ be the angle which the rod makes with the horizontal. Taking O as origin, the y-axis downwards, and the x-axis horizontal and towards the broken string, the co-ordinates of G are

$$x = a\cos\theta - l\sin\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$
$$y = a\sin\theta + l\cos\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Neglecting squares of $\frac{d\theta}{dt}$ and $\frac{d\phi}{dt}$, which are initially zero, we find on differentiating (1) and (2),

$$\frac{d^2x}{dt^2} = -a\sin\theta\frac{d^2\theta}{dt^2} - l\cos\phi\frac{d^2\phi}{dt^2} \quad . \quad . \quad . \quad . \quad (3)$$

$$\frac{d^2y}{dt^2} = a\cos\theta\frac{d^2\theta}{dt^2} - l\sin\phi\frac{d^2\phi}{dt^2} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

But since we want the initial accelerations only we must use the initial values of θ and ϕ, namely, $\theta = 0$ and $\phi = \phi'$ say. Thus the initial accelerations are

$$\frac{d^2x}{dt^2} = -l\cos\phi'\frac{d^2\phi}{dt^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\frac{d^2y}{dt^2} = a\frac{d^2\theta}{dt^2} - l\sin\phi'\frac{d^2\phi}{dt^2}$$

$$= a\frac{d^2\theta}{dt^2} - a\frac{d^2\phi}{dt^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Hence the equations of motion are

$$-Ml\cos\phi'\frac{d^2\phi}{dt^2} = T\sin\phi' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$Ma\left(\frac{d^2\theta}{dt^2} - \frac{d^2\phi}{dt^2}\right) = Mg - T\cos\phi' \quad . \quad . \quad . \quad . \quad (8)$$

$$\tfrac{1}{3}Ma^2\frac{d^2\theta}{dt^2} = aT\cos\phi' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Substituting in (8) the values of $\frac{d^2\theta}{dt^2}$ and $\frac{d^2\phi}{dt^2}$ from (7) and (9), we get

$$3T\cos\phi' + \frac{a}{l}T\tan\phi' = Mg - T\cos\phi' \quad . \quad . \quad . \quad (10)$$

or $$T(4\cos\phi' + \sin\phi'\tan\phi') = Mg \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Therefore $$T = Mg\frac{\cos\phi'}{3\cos^2\phi' + 1} \quad . \quad . \quad . \quad (12)$$

Before the string broke the tension in each was

$$T_0 = \tfrac{1}{2}Mg\frac{1}{\cos\phi'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Hence $$\frac{T}{T_0} = \frac{2\cos^2\phi'}{3\cos^2\phi' + 1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

We can deduce the results for vertical strings by putting $\phi' = 0$ Then

$$\frac{T}{T_0} = \tfrac{1}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

In the foregoing solution the geometrical relation between x, y, and θ, has not been expressed in a single equation. It can be obtained by eliminating ϕ from (1) and (2). It is worth while to get this equation and solve the problem by a rather different method.

On eliminating ϕ from (1) and (2), we get

$$(x - a\cos\theta)^2 + (y - a\sin\theta)^2 = l^2 \quad . \quad . \quad . \quad (16)$$

Let us write h for the equilibrium value of y, namely, $l\cos\phi'$ or $\sqrt{l^2 - a^2}$. Then put

$$y = h + z \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Now x, z, and θ, are small quantities. Consequently we need only retain the first powers of these quantities in (16). Substituting for y in (16), and then retaining only first powers of x, z, and θ, we get

$$-2ax + 2hz - 2ha\theta = 0 \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Removing the factor -2, and then differentiating twice

$$a\frac{d^2x}{dt^2} - h\frac{d^2z}{dt^2} + ha\frac{d^2\theta}{dt^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad (19)$$

There is still another way of arriving at equation (19). The geometrical equation (16) expresses the fact that the length OA is constant. We may now get equation (19) from the consideration that A moves perpendicular to OA, that is, that the component velocity of A along OA is zero. This method we shall now use.

The velocity of A is the resultant of the velocity of G and its velocity relative to G. But at the beginning of the motion the velocity of A relative to G was merely $a\frac{d\theta}{dt}$ vertically upwards. Hence the initial horizontal and vertical velocities of A were $\frac{dx}{dt}$ and $\left(a\frac{d\theta}{dt} - \frac{dy}{dt}\right)$ respectively. Resolving these along AO and writing $\frac{dz}{dt}$ for its equivalent $\frac{dy}{dt}$, we get

$$\sin\phi\frac{dx}{dt} + \cos\phi\left(a\frac{d\theta}{dt} - \frac{dz}{dt}\right) = 0 \quad . \quad . \quad . \quad . \quad (20)$$

Differentiating this with respect to t and neglecting $\frac{d\phi}{dt}$, which is initially zero, we get

$$\sin\phi\frac{d^2x}{dt^2} + \cos\phi\left(a\frac{d^2\theta}{dt^2} - \frac{d^2z}{dt^2}\right) = 0 \quad . \quad . \quad . \quad (21)$$

If we put ϕ' for ϕ in this and then multiply by l, we shall get the same equation as (19).

P be any particle of the body which is, at the instant considered, in the plane AOB. We shall show that the rotations are equivalent to a rotation about OC, and represented by OC. Let p and q be the lengths of the perpendiculars from P on OA and OB.

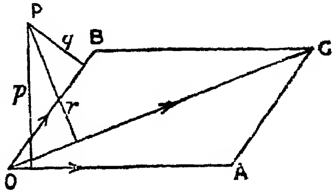


FIG. 206.

After an interval of time dt, P is raised perpendicularly out of the plane AOB by an amount $(p\alpha + q\beta)dt$ due to the two rotations; and this will be true for all positions of P, provided we consider the perpendicular on either of these vectors to be positive or negative according as P is on the left or right of the vector. In the position shown in the figure both perpendiculars are positive. But $p\alpha + q\beta$ is exactly the same in magnitude as the sum of the moments about P of forces represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$, and we have proved previously (Art. 60) that this sum is equal to the moment of the resultant $\overrightarrow{OC}$. That is,

$$p\alpha + q\beta = r \,.\, OC$$

where r is the length of the perpendicular from P on OC, with the same rule regarding its sign as we have given for the signs of p and q. Thus the displacement of P in the interval dt is $r \,.\, OC \,.\, dt$ perpendicular to the plane AOB, and this is exactly the displacement due to an angular velocity represented by $\overrightarrow{OC}$. Therefore the displacement of any point in the plane AOB due to the angular velocities $\overrightarrow{OA}$ and $\overrightarrow{OB}$ is the same as the displacement due to the angular velocity $\overrightarrow{OC}$. But the position of three non-collinear points of a rigid body will fix the position of the body. Consequently, since the particles in the plane of the axes are brought into the same position by the rotation about OC as by the rotations about OA and OB, it follows that the whole body is brought into the same position. That is, the angular velocity $\overrightarrow{OC}$ is the exact equivalent of the angular velocities $\overrightarrow{OA}$ and $\overrightarrow{OB}$. Thus angular velocities are added by vector rules.

By a successive application of the preceding argument it can be shown that the effect of any number of simultaneous angular velocities is the same as the effect of their vector sum.

If the point P lies on OC the perpendicular q is negative, and $p\alpha + q\beta$ is zero. This shows that OC is at rest, and proves more directly that OC is the axis of rotation.

The axes OA and OB need not be fixed in space. They need only be instantaneous axes of rotation. If OA and OB are not fixed in space, then OC will not generally be fixed.

The following is an example of a body rotating about two axes at the same time.

A top is spinning about its axis of figure, which is inclined to the vertical at an angle a, and rotating about that vertical. Suppose the direction of spin to be such that the vector representing the angular velocity is drawn upwards along the axis. Suppose also that the axis of the top itself is rotating about the vertical with an angular velocity ω in the direction represented by an upward vector. If n is the angular velocity of the top about its axis, the horizontal and vertical components of the total angular velocity of the top are $n \sin a$ and $n \cos a + \omega$. Hence the resultant angular velocity is

$$\sqrt{\{n^2 \sin^2 a + (n \cos a + \omega)^2\}} = \sqrt{\{n^2 + \omega^2 + 2n\omega \cos a\}} \quad (1)$$

and the axis of this rotation makes an angle θ with the vertical given by

$$\tan \theta = \frac{n \sin a}{n \cos a + \omega} \quad \ldots \ldots \quad (2)$$

and it is in the vertical plane containing the axis of the top.

If ω and n are constant, the axis of the top and the axis of resultant angular velocity describe cones about the vertical axis.

468. Moment of a Vector about any Axis.—Vectors which are situated in definite lines, such as forces, momenta, and accelerations, have moments about any axis. We have already had examples of moments of forces and moment of momentum, but in these cases the axes were always perpendicular to the planes of the vectors. We are now about to extend the definition of the moment of a vector.

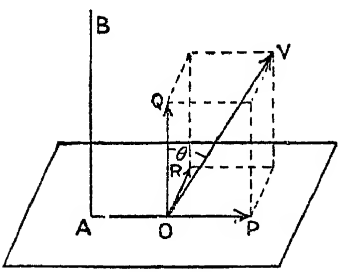


FIG. 207.

Let $\overrightarrow{OV}$ be any vector in space and AB any line. From any point O on the line of the vector the perpendicular OA is drawn to the line AB. The vector $\overrightarrow{OV}$ is resolved into three components $\overrightarrow{OP}$, $\overrightarrow{OQ}$, $\overrightarrow{OR}$, the first along AO, the second parallel to AB, and the third perpendicular to the plane of the other two. Then the moment of $\overrightarrow{OV}$ about AB may be defined as the product of AO and the component $\overrightarrow{OR}$. In the language of the earlier part of this book it is the moment of the vector $\overrightarrow{OR}$ about the point A. This is, of course, the same thing as the moment of the resultant of $\overrightarrow{OP}$ and $\overrightarrow{OR}$ about A. If p is the length of the common perpendicular to AB and OV, produced if necessary, it is easy to show that the moment defined is $p \cdot \text{OV} \sin \theta$; for the resultant of $\overrightarrow{OP}$ and $\overrightarrow{OR}$ is $\text{OV} \sin \theta$, and the

perpendicular from A on this resultant vector is the perpendicular from A on the plane QOV which is equal to the common perpendicular p between OV and AB, from which it follows that the moment of OV $\sin\theta$ about A is p . OV $\sin\theta$.

In the next article we shall show that the moments of a given vector about different axes passing through one point may be regarded as the components along these axes of a certain vector which we shall call the resultant moment for that point.

469. Moments of a Given Vector about Different Axes through a Given Point.—Let A be the given point and V the given vector, and let the perpendicular distance AP from A on V be denoted by p.

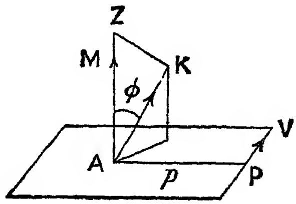


FIG. 208.

Let two equal and opposite vectors, V and $-$V, be introduced at A, each parallel to the given vector V, just as was done with forces in Art. 71. By analogy with statics we shall call the pair of vectors, V at P and $-$V at A, a *couple;* if we are dealing with forces it will be a *force-couple,* if with momentum it will be a *momentum-couple.* We could likewise have velocity and acceleration couples if we needed them. Then the original vector V at P is equivalent to an equal vector V at A, together with a couple which can be represented by a vector of magnitude $M = pV$ along the normal AZ to the plane of the couple, that is, the plane containing the point A and the vector V. This method of representing couples has already been explained in Art. 67.

We can get all our rules for couples of any kind from the rules for force-couples, since the proofs are the same for all couples. For example, the component of the couple M about any line AK inclined at an angle ϕ with AZ is $M\cos\phi = pV\cos\phi$, the other component of M being at right angles to AK. Now, the component $M\cos\phi$ must be the moment of V about AK: for the vector V at P can be resolved into components respectively parallel and perpendicular to AK, and these can be transformed, like the vector V itself, into a pair of vectors at A together with a pair of couples. Clearly the pair of vectors at A must have a resultant V, and the pair of couples obtained in this way must have a resultant M. It is clear also that the vectors of this pair of couples are respectively perpendicular and parallel to AK, and the one parallel to AK must be the component we obtained previously, the magnitude of which was $pV\cos\phi$. But the moment of this latter couple is (by the definition given in the last Art.) the moment of the vector V about AK. Thus we have arrived at the conclusion that the moment of a given vector V about any axis AK through a fixed point A is the component along AK of the moment of the couple formed by the given vector V and another vector $-$V acting at the point A. This last couple, of magnitude pV, we may regard as the resultant moment of the vector V about the point A.

As an illustration of the theory of this article, of which we have

already made use, we refer to the beginning of Chapter III., where we replaced a force F, which had components X, Y, Z, acting at the point $x. y, z$, by a force F at the origin O and three component couples of moments $(yZ - zY)$, $(zX - xZ)$, $(xY - yX)$. These three component couples are the moments of the given force F about OX, OY, OZ respectively.

The resultant couple at O is what we may call the resultant moment at O of the force F. This resultant moment is, as we see, really the moment of a force-couple when F is a force, and it would be the moment of a momentum-couple if the vector F were the momentum of a particle passing through the point x, y, z.

For a given vector F acting along a fixed line we see that there is a definite couple vector associated with every point A in space. For the general vector V the couple vector has a magnitude pV, and its direction is perpendicular to the plane of A and the vector V, and this vector we have decided to call the resultant moment of V about A.

470. Moment of any Number of Vectors.—If any number of vectors V_1, V_2, etc., of the same sort, acting along fixed lines, be each replaced by an equal vector at A and its associated couple C_1, C_2, etc., the resultant of the couple-vectors is a single couple C, which is the vector sum of the moments of all the vectors about A, and may therefore be regarded (or rather defined) as the resultant moment of all the vectors at A. So the original vectors V_1, V_2, etc., can be replaced by a single vector V at A (which is the resultant of the vectors V_1, V_2, etc., acting at A) together with a single couple C. We have already shown how that can be done analytically for forces in Art. 82. Since the sum of the moments of all the vectors about any axes through A is the sum of the components of C_1, C_2, etc., along this axis, it follows by the method of Art. 38 that this sum is correctly given by the component of the couple vector C along this axis.

To recapitulate: Any number of vectors in space are equivalent to a single resultant vector of the same sort at any given point A together with the couple-vector C associated with A. The sum of the moments of all the vectors about any axis through A is merely the component of C along that axis.

471. The Momentum Vector of a Rigid Body.—Let a particle A of a rigid body have component velocities u_1, u_2, u_3 parallel to the three co-ordinate axes AX, AY, AZ; and let the component angular velocities of the rigid body about these axes be ω_1, ω_2, ω_3, the resultant being $\Omega = \sqrt{\omega_1{}^2 + \omega_2{}^2 + \omega_3{}^2}$. Also let $\bar{x}$, $\bar{y}$, $\bar{z}$, be the co-ordinates of the centre of mass of the body. Then the velocity of the particle m situated at x, y, z is the vector sum of the velocity of A and the velocity due to the rotations. Let us consider first the z-component velocity of the particle which is situated at the point (x, y) in the x-y plane. Due to the rotations this velocity is clearly $y\omega_1 - x\omega_2$, and so its whole z-component velocity is $u_3 + y\omega_1 - x\omega_2$. But every particle of a rigid body which lies on a line parallel to the z-axis at any instant must have the same component velocity in the z-direction;* otherwise the particles would not remain at the same distance apart, and the

body is not a rigid body unless the particles do remain at the same distance apart. It follows, then, that the particle m, situated at x, y, z, has the same component velocity in the z-direction as the particle at (x, y, o), namely, $u_3 + y\omega_1 - x\omega_2$. Hence the components of the momentum of this particle are seen, by cyclic changes, to be

$$m(u_1 + z\omega_2 - y\omega_3),\ m(u_2 + x\omega_3 - z\omega_1),\ m(u_3 + y\omega_1 - x\omega_2).$$

The moment of this momentum about the z-axis is

$$\begin{aligned} &xm(u_2 + x\omega_3 - z\omega_1) - ym(u_1 + z\omega_2 - y\omega_3) \\ &\qquad = m(xu_2 - yu_1) + \omega_3 m(x^2 + y^2) - \omega_1 mxz - \omega_2 myz. \end{aligned}$$

Therefore the total moment of the momentum of all the particles about AZ is

$$\begin{aligned} &\Sigma m(xu_2 - yu_1) + \omega_3\Sigma m(x^2 + y^2) - \omega_1\Sigma mxz - \omega_2\Sigma myz \\ &\qquad = \bar{x}Mu_2 - \bar{y}Mu_1 + I_3\omega_3 - I_{13}\omega_1 - I_{23}\omega_2 \quad . \quad (1) \end{aligned}$$

where the I's denote moments and products of inertia and M is the total mass.

There are two simple cases in which this last expression reduces to the terms containing the angular velocities only. They are

(i) when the particle A is at rest, whether instantaneously or permanently; for then $u_1 = u_2 = o$; and

(ii) when A is the centre of mass of the body, whatever be its velocity, for then $\bar{x} = \bar{y} = o$.

In these two cases the components of the momentum couple of the whole rigid body for the point A about the three axes reduce to

$$\left.\begin{aligned} &(I_1\omega_1 - I_{12}\omega_2 - I_{13}\omega_3) \\ &(I_2\omega_2 - I_{23}\omega_3 - I_{12}\omega_1) \\ \text{and}\quad &(I_3\omega_3 - I_{13}\omega_1 - I_{23}\omega_2) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

respectively.

If the axes AX, AY, AZ, are the principal axes of inertia at A, then $I_{12} = I_{23} = I_{13} = o$, and consequently, in the two cases just mentioned, the components of the momentum couple associated with A still further reduce to $I_1\omega_1$, $I_2\omega_2$, and $I_3\omega_3$ respectively. Then the resultant couple is

$$C = \sqrt{I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2} \quad . \quad . \quad . \quad . \quad (3)$$

and the vector representing this couple makes with the principal axes angles α, β, γ, given by

$$C\cos\alpha = I_1\omega_1,\ C\cos\beta = I_2\omega_2,\ C\cos\gamma = I_3\omega_3 \quad . \quad (4)$$

It should be remarked that the axis of the couple C does not usually coincide with the axis of resultant angular velocity Ω, for this latter axis is inclined to the principal axes at angles α', β', γ', given by

$$\Omega\cos\alpha' = \omega_1,\ \Omega\cos\beta' = \omega_2,\ \Omega\cos\gamma' = \omega_3 \quad . \quad . \quad (5)$$

Only if $I_1 = I_2 = I_3$ could the directions of Ω and C coincide.

Having found the momentum couple we shall now find the momentum vector at A.

The x-component of the momentum vector at A is

$$\left.\begin{aligned}\Sigma m(u_1 + z\omega_2 - y\,\omega_3) &= \mathrm{M}u_1 + \omega_2\Sigma mz - \omega_3\Sigma my \\ &= \mathrm{M}u_1 + \mathrm{M}\omega_2\bar{z} - \mathrm{M}\omega_3\bar{y}\end{aligned}\right\} \qquad (6)$$

Now let us assume for the moment that A coincides with the centre of mass, and let us write $\bar{u}_1$ for u_1 in this case. Then the above component of the momentum reduces to $\mathrm{M}\bar{u}_1$. Likewise the other two components are $\mathrm{M}\bar{u}_2$ and $\mathrm{M}\bar{u}_3$. If, then, V denotes the velocity of the centre of mass it is clear that the resultant momentum vector is MV and is in the direction of V. It is also clear that this resultant is independent of the position of A, because the magnitudes and directions of the vectors were not altered when they were moved to A in the summing process.

It is now easy to see that the simplest representation of the momentum vector is by means of a vector MV acting at the centre of mass G together with the couple C given by (3). To get the momentum couple for any other point B we need only add the couple C to the couple obtained by shifting MV from G to B. If this resultant couple vector is C_{B}, then the moment of the momentum of the whole rigid body about any axis through B is given by the component of C_{B} along that axis.

While, of course, we could use the method described in the last paragraph to get the momentum couple for a *fixed* point A of the body, it is, nevertheless, easier to get it directly by using the principal axes at A. The momentum couple for the point A is similar to that for the point G, but it involves the principal axes and principal moments of inertia at A instead of the corresponding principal axes and principal moments of inertia at G.

If the centre of mass G is at rest it should be noticed that the momentum vector through G vanishes, leaving only a momentum couple, which has the same moment about all parallel axes.

472. One of the fundamental principles in the dynamics of a particle moving in one plane is that the rate of increase of the moment of momentum of the particle about any point in the plane is equal to the moment about that point of the forces acting on the particle. We shall now extend this principle to three-dimensional motion.

If the co-ordinates, at any instant, of a particle of mass m referred to three mutually perpendicular axes are x, y, z, and if forces whose components parallel to the axes are X, Y, Z, act on the particle, then

$$m\frac{d^2x}{dt^2} = \mathrm{X} \qquad (1)$$

$$m\frac{d^2y}{dt^2} = \mathrm{Y} \qquad (2)$$

$$m\frac{d^2z}{dt^2} = \mathrm{Z} \qquad (3)$$

Multiplying both sides of (1) by $-y$ and both sides of (2) by x, and adding the corresponding sides of the resulting equations, we get

$$m\left(x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2}\right) = x\mathrm{Y} - y\mathrm{X} \quad . \quad . \quad . \quad (4)$$

This can be written

$$\frac{d}{dt}\left(xm\frac{dy}{dt} - ym\frac{dx}{dt}\right) = x\mathrm{Y} - y\mathrm{X} \quad . \quad . \quad . \quad . \quad (5)$$

The right-hand side of (5) is the moment of all the forces acting on the particle about the axis of z; and the left-hand side is the rate of increase of the moment of the momentum of the particle about the same axis. Since the z-axis can be in any position we choose, equation (5) shows, therefore, that *the rate of increase of the moment of momentum of a particle about any fixed axis is equal to the moment about the same axis of the forces acting on the particle.*

473. If we are dealing with several particles, whether connected or not, we should get an equation similar to (5) for each particle. By summing all these equations, we get

$$\frac{d}{dt}\sum\left(xm\frac{dy}{dt} - ym\frac{dx}{dt}\right) = \Sigma(x\mathrm{Y} - y\mathrm{X}) \quad . \quad . \quad . \quad (1)$$

The moments of the interactions between the particles will disappear from the right-hand side and only the moment of the external forces will be left. The left-hand side of (1) is the rate of increase of the moment of momentum of the whole system of particles about the axis of z. It follows, therefore, that the principle stated at the end of the last article remains true if we replace the word "particle" by "system of particles." A system of particles includes a rigid body or several rigid bodies.

474. Change of Moment of Momentum of a System of Bodies about Axes through their Centre of Mass.—The theorem of the last article is true, not only for fixed axes, but for any axis through the centre of mass of the body or system of bodies with which we are dealing. The proof is exactly the same as in Art. 403.

If the system we are dealing with consists of several rigid bodies we need to know the moment of momentum of a rigid body about any axis. In Art. 471 we showed how to represent the momentum by means of a couple and a vector passing through G, and from this it follows that the moment of momentum of a rigid body about any axis is equal to the moment of momentum of a particle of the same mass moving with the centre of mass of the body together with the moment of momentum of the rigid body, due to its rotation only, about a parallel axis through its centre of mass.

We will now state the conclusion at which we have arrived.

The rate of increase of the moment of momentum of any system of bodies, either about a fixed axis in space or about an axis through the centre of mass of the system, is equal to the moment about the same axis of the external forces acting on the system.

475. If the external forces have no moment about the centre of mass, then the moment of momentum about any axis through the centre of mass remains unaltered. Thus, if we suppose that each member of

the solar system is acted on by no forces except the attractions of the other members, then the external forces on the system are zero, and therefore the moment of momentum of the whole system about any axis through the centre of mass remains constant. It follows, therefore, that the resultant moment of momentum at the centre of mass is constant in magnitude, and its axis has a fixed direction in space. The plane perpendicular to this axis through the centre of mass of the system has no motion in space except the uniform motion of the centre of mass. It is the nearest approach to a fixed plane that we know, and for this reason it is called *the invariable plane* of the solar system. It nearly coincides with the plane of Jupiter's orbit.

We may apply the principle of this article to the earth itself. The forces exerted on the earth pass very nearly through its centre of mass, the small deviations being due to the equatorial protuberance of the solid earth and to the tidal protuberances. It follows, then, that the moment of momentum of the earth about its axis of rotation at any instant remains nearly constant for a long period of time. The axis of rotation at any instant means, of course, a line whose direction is fixed in space and is the same as that of the axis of rotation at the given instant, and which always passes through the centre of mass. If the earth is shrinking its moment of inertia is diminishing, and therefore its angular velocity must be increasing to keep its moment of momentum constant.

476. When a body rotates about a fixed axis passing through its centre of mass, the student will probably jump to the conclusion that this axis must be the axis of resultant moment of momentum at the centre of mass. To avoid this error, and also to give a concrete example of moment of momentum, we shall consider the motion of a very simple rigid body about a fixed axis.

A rigid body, composed of two equal heavy particles and a rod of negligible mass joining them together, rotates about a fixed axis through the mid-point of the rod, so that the rod describes a cone and the particles describe equal circles.

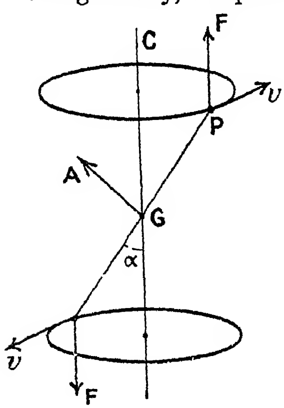


FIG. 210.

Let m be the mass of each particle, v its velocity. Then the momentum of each particle is mv. Let l be the length of the rod, r the radius of the circular path of a particle, and a the angle of the cone. Then the moment of momentum of each particle about the axis of rotation is rmv, and the moment of momentum of the rigid body about this axis is therefore $2rmv$. Now, it is obvious that the whole momentum reduces to a momentum-couple, the axis of which may be taken as GA, which is perpendicular to the rod and in the plane of the rod and the axis of rotation. The total moment of momentum about this axis GA

is lmv, and this is the resultant moment of momentum at G. If we resolve this along the axis of rotation we get

$$lmv \sin \alpha = 2 \cdot \text{GP} \cdot \sin \alpha \cdot mv = 2rmv,$$

which agrees with the result obtained by taking moments directly.

If v is constant the moment of momentum about the axis of rotation remains constant. But the component perpendicular to the axis of rotation, although its magnitude remains constant, rotates in the same time as the rod. The magnitude of the perpendicular component is $lmv \cos \alpha$; and if ω is the angular velocity of the rod, its rate of increase is $lmv \cos \alpha \cdot \omega$, by Art. 270. Moreover, the direction of this rate of increase is perpendicular to the vector $lmv \cos \alpha$ and in the plane in which the vector rotates, that is, it is perpendicular to the plane PGC. But the rate of change of moment of momentum about any axis is equal to the moment of the forces about the same axis. There must then be some forces acting on the system which have a moment about the line through G perpendicular to the plane PGC. These forces might be applied to the particles themselves. If a downward force were applied to the lower particle and an upward force to the upper particle, the system would be exactly similar to two conical pendulums one of which is upside down. If F is the force applied to each particle, the principle of Art. 474 gives

$$2r\text{F} = lmv\omega \cos \alpha \qquad (1)$$

But $v = r\omega$. Hence

$$\text{F} = \tfrac{1}{2}m\omega^2 l \cos \alpha \qquad (2)$$

Writing h for $\frac{1}{2}l \cos \alpha$, the height of each cone,

$$\text{F} = mh\omega^2 \qquad (3)$$

This is exactly what we should get by considering the circular motion of each particle, dealing with it in the same way as we deal with a conical pendulum.

477. A body with an axis of symmetry has a constant spin about this axis, and in addition this axis itself describes a circular cone at a constant rate about a fixed axis in space. To find the forces necessary to maintain this motion.

Let OP and OQ be the axes fixed in space and in the body respectively, θ the angle between them. Let the angular velocity of the body be equivalent to two components ω and n along OP and OQ respectively. The angular velocity of the plane POQ about OP is ω. The angular velocity n is not so easy to realize, and therefore we will state clearly what it is. When a body has simultaneous rotations about two axes, the motion of a particle on either axis is entirely due to rotation about the other. Consequently, the angular velocity about OQ

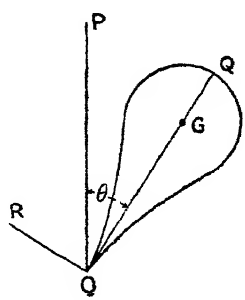


FIG. 211.

is the angle between two planes, fixed in the body and containing OQ, which pass across OP at intervals of a second.

Let A denote the moment of inertia of the body about OQ, B its moment of inertia about an axis through O perpendicular to OQ.

Let OR be taken perpendicular to OQ in the plane POQ. Then the angular velocity ω may be resolved into two components, $\omega \cos \theta$ along OQ and $\omega \sin \theta$ along OR. Thus there is $(n + \omega \cos \theta)$ along OQ, and $\omega \sin \theta$ along OR, and these are equivalent to the given angular velocities.

Since OQ and OR are principal axes at O the results of Art. 471 show that the components of moment of momentum about these axes are $A(n + \omega \cos \theta)$ and $B\omega \sin \theta$. Since there is no angular velocity about the third principal axis it follows that the horizontal component of the moment of momentum is

$$A(n + \omega \cos \theta) \sin \theta - B\omega \sin \theta \cos \theta \quad . \quad . \quad . \quad . \quad (1)$$

in the direction of the projection of $\overrightarrow{OQ}$. This vector rotates with an angular velocity ω, and therefore its rate of increase is the product of the vector and ω. The direction of the vector representing the rate of increase is parallel to the velocity of G, and its axis is at O. There must then be forces acting on the body, whose moment about this axis through O is

$$\omega\{A(n + \omega \cos \theta) \sin \theta - B\omega \sin \theta \cos \theta\} \quad . \quad . \quad (2)$$

in such a direction that θ would increase if the body yielded to the forces, as it would do but for its rotation.

If θ is a right angle, this moment is $An\omega$; and if θ is zero, the moment is zero, as we might expect. Also the moment is constant if θ is constant.

If r is the distance of the centre of mass of the body from OP, there is also a force $Mr\omega^2$ parallel to the perpendicular from G on OP. This is the force necessary to make G describe its circular path.

478. Motion of a Top under its own Weight.—If the only forces acting on the body considered in the last article are its weight parallel to PO and the reaction of the support at O, then the moment of the weight causes the rotation of the axis. Equating this moment to the rate of change of moment of momentum, we get

$$Mgh \sin \theta = \omega\{A(n + \omega \cos \theta) \sin \theta - B\omega \sin \theta \cos \theta\} \quad . \quad (1)$$

where $h = OG$.

Hence $$Mgh = \omega\{(A - B)\omega \cos \theta + An\} \quad . \quad . \quad . \quad . \quad (2)$$

This gives ω when n and θ are given. If $A = B$, or if $\theta = \frac{\pi}{2}$, the relation (2) becomes

$$Mgh = An\omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In general there are two possible values of ω for given values of n and θ. If a top is set spinning, and then left to itself with the end of its axis fixed, the motion of the axis is at first seemingly irregular, but it finally settles down to the sort of steady motion we have just been considering; and when this steady motion does occur the value of ω has always the same sign as n, and is usually small compared with n. The actual value of ω is, in fact, approximately that obtained by neglecting the term containing ω^2 in (2). Thus

$$\omega = \frac{Mgh}{An} \text{ nearly}$$

$$= \frac{Mgh}{Mk^2n} = \frac{gh}{k^2n} \quad \ldots \ldots \ldots \quad (3)$$

where k is the radius of gyration about the axis of symmetry.

For a small gyroscope top the following are rough values for the quantities on which ω depends:—

$$h = 1\tfrac{1}{2} \text{ inches}, \ k = 1 \text{ inch} \quad \ldots \ldots \quad (4)$$

If the top is spinning at 100 revolutions per second about its axis, then $n = 200\pi$. Therefore

$$\omega = \frac{h}{k} \cdot \frac{g}{200\pi k} = \frac{3}{2} \cdot \frac{32 \times 12}{200\pi} = 0{\cdot}916 \quad \ldots \quad (5)$$

and the axis makes a revolution about the vertical in

$$\frac{2\pi}{\omega} = 6{\cdot}85 \text{ secs.} \quad \ldots \ldots \ldots \quad (6)$$

To estimate the insignificance of the term neglected in (2) for the particular problem just considered, we have only to notice that $(A - B)\omega \cos\theta$ was neglected in comparison with An. The magnitude of the ratio of the former to the latter quantity will be something in the neighbourhood of $\frac{\omega}{n} \cos\theta$, which is less than $\frac{1}{600} \cos\theta$.

The steady motion of the axis of the top on a cone is called *precession*, and we see from what precedes that this motion is possible under a constant force acting at a fixed point of a body spinning with one point of its axis fixed. But although it is possible, it is not the only possible motion, for there are an infinite number of modes of motion of the spinning top. But the steady motion considered is very nearly the actual motion of the top, however it is started, provided it has sufficient spin after frictional resistances have damped out the apparent irregularities.

479. Motion of a Spinning Body under no Forces.—If the fixed point of the body considered in Art. 477 coincides with the centre of gravity, the resultant force on the body must be zero. The only action on the body will therefore be the couple whose moment is given in equation (2) of that article. Denoting this moment by N,

$$N = \omega \sin\theta \, \{A(n + \omega\cos\theta) - B\omega\cos\theta\} \quad \ldots \quad (1)$$

If this couple is zero, there are no forces whatever acting on the body. In this case the motion could be maintained if the body were left entirely to itself after being once started. To make N zero we must have

$$A(n + \omega \cos \theta) - B\omega \cos \theta = 0 \quad . \quad . \quad . \quad . \quad (2)$$

that is
$$\omega \cos \theta = \frac{An}{B - A} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Thus ω and n have the same or different signs according as B is greater or less than A. For a disc such as a penny, $A = 2B$, so that

$$\omega \cos \theta = -2n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The instantaneous axis of rotation always lies in the plane of the symmetrical axis and the fixed axis OP (Fig. 211). Consequently, this instantaneous axis rotates about OP with the same velocity as OQ, that is, ω. The resultant angular velocity about the instantaneous axis is

$$\sqrt{(n^2 + \omega^2 + 2n\omega \cos \theta)} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The instantaneous axis always makes an angle ϕ with the axis of symmetry such that

$$\tan \phi = \frac{\omega \sin \theta}{\omega \cos \theta + n}$$

$$= \frac{A}{B} \tan \theta \text{ by } (2) \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Thus if a body with a symmetrical axis is set spinning about a line inclined at an angle ϕ with its symmetrical axis, and is then acted on by no forces, or only by forces at its centre of gravity, this symmetrical axis will describe a cone about a line which has a fixed direction in space and is inclined to the symmetrical axis at an angle θ given by (6). This fixed line in space is the axis of resultant moment of momentum, and we shall refer to it as the fixed axis.

The instantaneous axis of rotation describes a cone in the body itself, since it always makes an angle ϕ with the symmetrical axis. Now any plane in the body containing the symmetrical axis passes through the fixed axis once in every $\frac{2\pi}{n}$ seconds. And, since the instantaneous axis is always in the plane containing the fixed axis and the symmetrical axis, it follows that the instantaneous axis returns to the same position in the body once in every $\frac{2\pi}{n}$ seconds.

When the angular velocity of the body is expressed as two components, one along and one perpendicular to its axis, the component along its axis is $(n + \omega \cos \theta)$. Calling this s, equation (2) gives

$$\omega \cos \theta = \frac{A}{B} s \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

480. Free Motion of the Earth's Axis.—The earth is a body with a symmetrical axis which very nearly coincides with the axis of rotation. Owing to the equatorial protuberance of the earth the sun's and moon's attractions do not exactly pass through its centre of mass. The moments of these attractions about the centre of mass cause the earth's axis to describe a cone about the normal to the ecliptic in a period of about 25,700 years. This motion of the axis is called the precession of the axis.

But besides this very slow motion, which we may neglect for the present, there is very much quicker motion of the axis, the free motion investigated in the last article. This free motion is a consequence of the fact that the instantaneous axis of rotation does not coincide with the symmetrical axis. That this free motion does exist is considered to be established by a certain periodic variation of observed latitudes which is believed to have been discovered. The latitude of a place, as given by observations, is the complement of the angle between the axis of rotation and the radius of the earth through that place. Then, since the axis of rotation describes a cone in the earth with semi-angle ϕ, it follows that the largest and smallest observed latitudes at any place on the earth will differ by 2ϕ, the whole angle of the cone. That is, there is a periodic variation of all latitudes of amplitude ϕ, and the period of this variation is the period in which the instantaneous axis of rotation describes its cone in the earth's body. This period is, as proved in the last article, $\frac{2\pi}{n}$; and it is stated that observations give $\phi = 0{\cdot}148''$.

Now, by the theory of the precession of the earth's axis and observations on its motion, it is found that

$$\frac{A}{A - B} = 310 \text{ about.}$$

But
$$\tan\phi = \frac{A}{B}\tan\theta, \text{ by (6), Art. 479,}$$

or, since the angles are very small,

$$\phi = \frac{A}{B}\theta$$

Thus θ is very slightly less than ϕ.

By the last article

$$\omega\cos\theta = \frac{An}{B - A}$$

On account of the extreme smallness of θ we may take $\cos\theta = 1$. Then we get

$$n = -\frac{A - B}{A}\omega = -\frac{1}{310}\omega$$

The difference in sign of $\omega\cos\theta$ and n only means that the rotations are in opposite directions; that is, the vectors representing ω and n make an obtuse angle with each other.

Let ω_1 denote the resultant angular velocity of the earth. Then

$$\omega_1{}^2 = \omega^2 + n^2 + 2n\omega \cos \theta$$
$$= (\omega + n)^2 \text{ nearly}$$
$$\omega_1 = \omega + n$$
$$= (-310 + 1)n$$

Therefore $$n = -\tfrac{1}{309}\omega_1$$

Now the period of the variation of latitudes should be exactly

$$\frac{2\pi}{-n}, \text{ which} = 309\frac{2\pi}{\omega_1} \text{ nearly}$$
$$= 309 \text{ days.}$$

Since the instantaneous axis is always in the same plane as the fixed axis and the axis of figure, it follows that the period in which the instantaneous axis describes its cone in space is the same as the period in which the axis of figure describes its cone in space, and this is

$$\frac{2\pi}{\omega} = -\frac{2\pi}{310n}$$
$$= \tfrac{309}{310} \text{ of a day.}$$

Also a figure will show that the resultant ω_1, of n and ω, lies beyond the fixed axis from the axis of figure and makes an angle $(\phi - \theta)$ with the fixed axis. This is an extremely small angle, being only about $0{\cdot}0005''$.

But the observed period of variation of latitudes is about 428 days. The discrepancy is supposed to be due to the elastic yielding of the earth. The instantaneous axis does not coincide with the axis of figure, for the rotation about the instantaneous axis causes the earth to change its shape in such a way that the axis of the equatorial bulge is nearer the axis of rotation at any instant than it would be if the earth were perfectly rigid.

481. General Motion of a Spinning Top.—It was shown in Arts. 477 and 478 that a possible mode of motion of a spinning top is one in which the spin remains constant and the axis of symmetry of the top describes a cone about the vertical at a uniform rate. But it is obvious that the top will have to be started in a particular way in order that it may assume this motion. It is true that, in most actual cases, friction eventually reduces the motion to the sort of motion considered in Art. 478. But we now want to consider the motion when friction does not act, or before friction has had time to destroy the effect of the initial motion of the symmetrical axis.

Firstly, let us examine the motion supposing the weight did not act. If the end of the axis of the top is fixed and no forces act except the reaction at the point of support, the resultant moment of momentum at that point will remain constant. If A and B are the principal moments

of inertia at the end of the axis, it may be shown, exactly as in Art. 479, that the symmetrical axis of the body will generally rotate about some axis through the fixed point with an angular velocity ω_1 given by

$$\omega_1 \cos \theta_1 = \frac{A}{B} s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This is the same equation as (7), Art. 479, and the only difference is that here it is not the centre of gravity but the end of the axis that is fixed.

Here θ_1 means the semi-vertical angle of the cone described by the axis of the top. If this angle is small, we may take

$$\omega_1 = \frac{A}{B} s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Now, the actual motion is a combination of the precessional motion of Art. 478 and the free motion just explained. If the top is started in such a way that θ_1 is small, the axis of the top describes a cone of semi-vertical angle θ_1 about an axis which itself describes a cone about the vertical. We may explain the motion in another way by stating that the upper end of the axis describes a small circle on a sphere about a centre which itself describes a horizontal circle. Thus the end of the axis describes a sort of epitrochoid on the sphere. This is the usual motion of the axis when a top is started in any manner.

Suppose the top is started with its axis at rest at any inclination α to the vertical. If we think of the axis of the top as rigidly attached to the small cone of semi angle θ_1, it is clear that this cone must be rolling on the cone which has a vertical axis and semi-angle α, for in no other way could the axis of the top be at rest. Or, from the point of view of the circles on the sphere, the free end of the axis of the top lies on the circumference of the circle which rolls on the horizontal circle whose radius subtends an angle α at the centre. Thus the epitrochoid becomes, in this particular case, an epicycloid. If α is approximately a right angle, the portion of the sphere on which the end of the axis moves is nearly cylindrical, so that the epicycloid is approximately a cycloid on the flattened-out surface of this portion of the sphere.

If ω is the precessional angular velocity of the axis about the vertical, equation (3), Art. 478, gives

$$\omega = \frac{Mgh}{An} \text{ nearly} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

But here n is very nearly the same thing as s in equation (2) above. For the quantity in Art. 478 which corresponds to s is $(n + \omega \cos \theta)$, and ω is usually small compared with n. Hence we find

$$\omega = \frac{Mgh}{As} \text{ nearly} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

482. **Motion of a Spinning Projectile.**—An elongated projectile fired from a rifled gun emerges with a spin about its long axis. If it

were not for the resistance of the air it would continue to spin about this axis, and the direction of the axis would remain fixed in space. Now in consequence of the downward curvature of the path of the projectile, the axis of the shot is soon pointing in a direction above the direction of motion of the centre of gravity. It is well known that when any surface of a projectile is inclined to the direction of motion the pressure on the forward portions is greater than that on the hind portions. In consequence of this, the air exerts a resultant force on the shot which has a moment about the horizontal axis through the centre of gravity perpendicular to the direction of motion, the effect of which would be to turn the nose of the shot to a still greater inclination to the horizontal if it had no spin. But the effect on the spinning shot is to turn its axis about the line of motion in the same sort of way as the axis of a top turns about the vertical. If the spin of the shot is right-handed, the nose will begin to turn towards the right from the point of view of the gunner, and if the spin is left-handed, the nose will turn towards the left. If the nose is turned towards the right, the air-pressure on the left-hand side of the shot causes the centre of gravity to veer towards the right also. The centre of gravity of a spinning projectile does not, therefore, move in a vertical plane, but deviates to the right or left, according as it has a right-handed or left-handed spin.

A similar effect is produced on a disc, such as a penny, thrown into the air with a spin about its axis, the direction of motion of the centre of gravity being initially up the plane of the disc. If AB is the central section of the disc in the plane of motion, $\overrightarrow{MN}$ the direction of motion a short time after projection, the resultant of the air-pressure is at some point C above the centre of gravity of the disc. If the disc is thrown with the right hand and its spin is generated by rolling it along the finger as one throws it into the air, the spin and the initial moment of momentum are represented by vectors directed downward from the disc and perpendicular to it. Consequently, the moment of P, the resultant thrust, turns the moment of momentum vector towards the right at first. Then the greater pressure on the forward right side of the disc causes it to deviate towards the left from the point of view of the thrower.

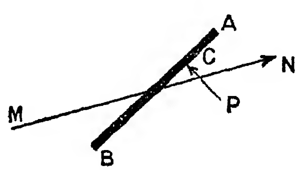


FIG. 212.

483. Rate of Change of Spin about the Symmetrical Axis.—We have shown that the rate of increase of a constant vector V, rotating with angular velocity ω about a line perpendicular to the vector, is $V\omega$ perpendicular to the plane containing the vector and the axis of rotation of the vector. If the vector V is not constant its rate of increase has, in addition, a component $\frac{dV}{dt}$ along the direction of the vector. This was proved in Art. 270, or it can be got by using the hodograph method.

Let G be the centre of mass of a body which has a symmetrical axis, and let GX, GY, GZ, be three mutually rectangular axes fixed in direction but following the centre of mass. Suppose GX coincides with the position of the symmetrical axis at a certain instant, and let the component angular velocities of the body at this instant be s about GX, η about GY, and zero about GZ. Then the component moments of momentum are As, Bη, and zero.

The vector As along GX rotates about GY, and therefore the component of its rate of increase due to the rotation of the vector is along GZ. Similarly, the component of the rate of increase of Bη due to its rotation about GX is along GZ. It is clear, then, that the only term in the rate of increase of the moment of momentum about GX is $A\frac{ds}{dt}$. If then H denotes the moment of all the forces about the symmetrical axis, it follows that

$$A\frac{ds}{dt} = H$$

and if H is zero for any interval of time, s must be constant during that interval.

It must be remembered that s is not the n of the previous articles, for $s = n + \omega \cos\theta$.

EXAMPLES ON CHAPTER XXIII

1. A motor-car has a flywheel parallel to the wheels of the car. If the flywheel rotates in the same direction as the wheels show that its gyrostatic action would help the car to topple in turning a corner, whereas if it rotates in the opposite direction it helps to keep the car upright.

[The action on the car is the reaction to the couple on the fly wheel.]

2. A ship's flywheel has a mean radius of 6 feet and weighs 5 tons. Suppose the flywheel makes 80 revolutions per minute, the direction of rotation being such that the bottom of the wheel moves contrary to the ship's motion. What couple does the wheel exert on its bearings when the ship turns at a constant rate through 80° towards the right in one minute, and in what direction does it tilt the ship ?

[The couple is 1·10 in foot-ton units, and it tilts the ship towards the left.]

3. On the same axle are two similar wheels spinning with equal angular velocities in opposite directions. If the axle rotates with constant angular velocity about a line perpendicular to the axle and passing through its middle point, show that the wheels apply equal and opposite couples to the axis, and that no external force or couple is needed to maintain this motion.

4. A body which has a symmetrical axis spins about a smooth axle coinciding with this axis. The axle itself is rigidly attached to a rigid body which is free to rotate about a smooth fixed rod perpendicular to the

symmetrical axis. Show that the rotation of the rigid body about the rod under any forces is just the same as if the first body were not spinning.

[The rate of change of moment of momentum about the rod is equal to the moment of the forces about the rod. If I is the moment of inertia of the combined body about the rod, and ω its angular velocity, then the whole moment of momentum about the rod is $I\omega$ whether the first body is spinning or not. It follows that the spin of the first body does not affect ω nor its rate of change.]

5. A uniform disc of mass M and radius r spins with angular velocity n about a thin smooth axle coinciding with the axis of the disc. The axle is free to turn in a vertical plane about a smooth horizontal rod passing through the axle at distance $2r$ from the centre of the disc. The disc is dropped from the position in which its axis is horizontal. Neglecting the mass of the axle, find the couple exerted on the rod when the axle makes an angle θ with the vertical.

[$2Mrn\sqrt{\tfrac{1}{17}gr\cos\theta}$ in foot-poundal units.]

6. It was once proposed to fit a cabin on a ship which should be free from one component of the angular motion of the ship, which we will call the rolling motion. The cabin was supported on a horizontal axis perpendicular to the plane of rolling, and a gyrostat, spinning about an axis perpendicular to the first axis, was rigidly attached to the cabin. This arrangement failed to steady the cabin. Why did it fail?

[The cabin turns about its axis just as freely when the gyrostat is spinning as when it is not spinning (see question 4). Consequently the gyrostat was useless.]

7. How must the ship's cabin mentioned in the last question be supported so that the gyrostat will keep it steady?

[It must be supported by one point, as, for example, by a chain attached above the centre of mass; or by a horizontal axis, as was proposed, attached to a frame which is itself free to turn about an axis perpendicular to the first axis. The essential thing is that the ship must not be capable of applying a couple to the cabin.]

8. Suppose that the mass of the rim of a bicycle wheel is M, and the total mass of its hub, spokes, and axle, is m; and assume that the moment of inertia of the hub and spokes is negligible compared with that of the rim. Let r denote the mean radius of the rim. One end of a string attached to a ceiling is fastened to a point on the axle at distance c from the centre of mass. If the wheel spins at n revolutions per minute, at what rate does it precess with the axle horizontal?

[The axle makes $\dfrac{900gc(M+m)}{\pi^2 r^2 n M}$ revolutions per minute.]

9. A top is formed out of a solid sphere of radius r with a thin axis through its centre, h being the distance of the end of the axis from the centre. If the top spins with angular velocity n on a smooth floor, at what rate does it precess with its axis inclined at a constant angle θ to the vertical?

[Since there are no horizontal forces on the top it is the centre of mass that remains fixed, not the point of support. At the centre of mass the principal moments of inertia are each $\frac{2}{5}Mr^2$. Therefore the equation

$$\{An + (A - B)\omega\cos\theta\}\omega\sin\theta = Mgh\sin\theta$$

gives
$$\tfrac{2}{5}Mr^2 n\omega = Mgh,$$

from which
$$\omega = \frac{5gh}{2r^2 n}\text{]}$$

10. Suppose the top mentioned in the last question were spinning with the end of the axis in a small smooth cup on the top of a fixed metal rod, and suppose that $r = 1$ inch, $h = \frac{4}{3}$ inches, $\theta = 60°$, and $n = 165$ radians per second. Calculate the two possible rates of precession.

[Assuming $g = 32$, the rates of precession are 48 and $10\frac{2}{3}$ radians per second.]

11. Show that the principal moments of inertia of a thin disc are equal at a point on its axis distant one half of the radius from the centre.

Suppose the disc spins about its axis at n radians per second with one of the two above-mentioned points fixed. At what rate will it precess about the vertical if its inclination is constant?

[The rate of precession is $\frac{g}{nr}$ radians per second, r being the radius of the disc.]

12. A top is made of a thin uniform disc of radius r and a rod of negligible mass along its axis. The top spins on a smooth floor with the axis inclined at the angle θ to the vertical, the point of support being at a distance $2r$ from the centre of the disc. If the top precesses steadily at ω radians per second, find the spin about its axis.

$$\left[\text{The spin } s = n + \omega \cos\theta = \frac{4g}{r\omega} + \tfrac{1}{2}\omega\cos\theta.\right]$$

13. A child's hoop rolls on a horizontal plane, the centre of the hoop moving with a speed v in a horizontal circle of radius r. Find the inclination of the plane of the hoop to the vertical.

What would be the inclination of a skater describing an equal circle with the same speed?

$$\left[\text{The inclinations are respectively } \tan^{-1}\frac{2v^2}{gr} \text{ and } \tan^{-1}\frac{v^2}{gr}.\right]$$

14. A uniform disc rolls on a horizontal plane, the centre describing a horizontal circle of radius r with a speed v. What must be the inclination of its plane to the vertical?

$$\left[\tan^{-1}\frac{3v^2}{2gr}.\right]$$

15. A body with a symmetrical axis is set in rotation with an angular velocity q about an axis inclined at an angle ϕ to the symmetrical axis, and is subsequently acted on by no forces. Show that its symmetrical axis will describe a cone with angular velocity ω about a line inclined at θ to the symmetrical axis, such that

$$\omega^2 = \frac{A^2\cos^2\phi + B^2\sin^2\phi}{B^2}q^2$$

and
$$\tan\theta = \frac{B}{A}\tan\phi$$

the angle θ being measured from the symmetrical axis towards the axis of q in the plane of both these axes.

16. A solid cylinder, whose length is equal to $\sqrt{3}$ times its radius, is thrown into the air spinning about an axis through its centre of mass. Show that the axis of rotation remains fixed in space and in the body except for the motion with the centre of mass, and that the angular velocity remains constant.

17. A penny is thrown into the air spinning with an angular velocity q about a line inclined at 30° to its symmetrical axis. Show that, while the

penny is in the air, its axis describes a cone of semi-angle 16° 6′ in the period $\frac{4\pi}{q\sqrt{13}}$, the vertex of the cone being at the centre of mass.

18. If A, B, C, are the principal moments of inertia of a rigid body at a point of the body which is kept fixed, and if α, β, γ, are its component angular velocities about the principal axes, prove that the kinetic energy of the body is

$$\tfrac{1}{2}A\alpha^2 + \tfrac{1}{2}B\beta^2 + \tfrac{1}{2}C\gamma^2$$

[If I is the moment of inertia about the instantaneous axis of rotation, and l, m, and n, the direction-cosines of this axis relative to the principal axes, then, q being the resultant angular velocity,

$$I = Al^2 + Bm^2 + Cn^2, \text{ by Art. 209;}$$

and $$\alpha = lq, \quad \beta = mq, \quad \gamma = nq.$$

The kinetic energy is clearly $\frac{1}{2}Iq^2$, which is equal to the given expression.]

19. If the angular velocity of a top or gyroscope is the resultant of ω about a vertical line and n about its axis, show that the minimum value of n for which steady precession is possible with the axis inclined at θ to the vertical is given by

$$A^2n^2 = 4(B - A)Mgh \cos\theta$$

where A and B denote the principal moments of inertia at the fixed point of the body. But if s denotes $(n + \omega\cos\theta)$, the component angular velocity about the symmetrical axis when the other component is perpendicular to it, show that the minimum value of s for steady precession at the same inclination is given by

$$A^2s^2 = 4BMgh\cos\theta$$

[Regard ω as an independent variable in both cases.]

20. If q is the minimum resultant angular velocity of a top for which steady precession is possible with the symmetrical axis inclined at θ to the vertical, show that

$$A^2q^2 = 2Mgh\cos\theta(\sqrt{A^2\tan^2\theta + B^2} + B)$$

21. A symmetrical flywheel can turn without friction about a diameter of a vertical metal ring, this diameter making an angle α with the vertical diameter. There is initially no rotation of the flywheel, but the ring is made to execute a complete revolution, from rest to rest, about its vertical diameter. Prove that the flywheel will have turned relatively to the ring through the angle $2\pi\cos\alpha$. *Manchester University, Honours Maths.*

[Since the forces acting on the flywheel have no moment about its axis its angular velocity about this axis remains zero. If θ is the angle through which the ring has turned at any instant, its component angular velocity about the axis of the flywheel is $\dot{\theta}\cos\alpha$. The relative angular displacement is thus $\theta\cos\alpha$, which becomes $2\pi\cos\alpha$ for a complete revolution.]

22. State carefully the principles of linear and angular momentum.

Assuming, in addition, the principle of energy, solve the following problem:—

The bob of a conical pendulum is held so that the string is tense and horizontal, and is projected horizontally at right angles to the string with the given velocity $\sqrt{2gc}$. Prove that when next it is moving horizontally the

velocity will be $\sqrt{2gy}$, where y is the positive root of the quadratic $y(y-c) = l^2$, and l denotes the length of the string.

Manchester University, Honours Engineering, Applied Maths. paper.

[Suppose z is the vertical fall of the body before the velocity becomes horizontal again, and let v be this velocity, and u the initial velocity. Then the energy equation gives

$$v^2 = u^2 + 2gz = 2g(c+z) = 2gy.$$

Since the forces on the bob have no moment about the vertical through the point of support the moment of momentum about this line remains constant. Therefore, equating the squares of the moment of momentum,

$$2gcl^2 = 2g(c+z)(l^2 - z^2) = 2g(cl^2 + zl^2 - z^3 - cz^2)$$

or $(z+c)z = l^2$;

i.e. $y(y-c) = l^2$.

23. A pendulum bob is projected in any manner. Show that, unless it is projected in the vertical plane containing the point of projection and the vertical through the point of support, the string can never be vertical.

24. Before the plug is pulled out of the bottom of a hemispherical wash-basin the water is set in rotation. Explain why the rotation increases as the basin empties.

[The moment of momentum of the whole mass about the vertical diameter is but slowly decreased by the resistance at the surface of the basin, and the water that escapes has little or no moment of momentum. Consequently the water that remains has a great part of the whole moment of momentum, which explains why its rotation increases.]

25. A body rotates with angular velocity q about an axis which is fixed in the body and in space. At a point on this axis the principal moments of inertia are A, B, and C. If the axis of rotation is in the plane of the axes of A and B and is inclined at the angle θ to one of them, show that the body exerts a couple on the axis whose magnitude is $(A - B)q^2 \sin\theta \cos\theta$.

26. If the body mentioned in the last question were spinning about a fixed axis whose direction-cosines were l, m, and n, referred to the axes of A, B, and C, show that the couple on the axis would be

$$q^2\sqrt{\{A^2l^2 + B^2m^2 + C^2n^2 - (Al^2 + Bm^2 + Cn^2)^2\}}.$$

[If R denotes the resultant moment of momentum, then the components of R along the principal axes are Alq, Bmq, and Cnq. Therefore

$$R^2 = (A^2l^2 + B^2m^2 + C^2n^2)q^2$$

Also, the component, P, of R along the axis of rotation is

$$P = (Al^2 + Bm^2 + Cn^2)q$$

The other component of R, namely, $\sqrt{(R^2 - P^2)}$, rotates in one plane with angular velocity q. Therefore its rate of increase is $\sqrt{(R^2 - P^2)}q$.]

CHAPTER XXIV

UNITS AND DIMENSIONS

484. THE fundamental units in physics and mechanics are the units of length, mass, and time. All other units may be derived from these three. The connection between force and these fundamental units is given by Newton's second law of motion, and the other units are connected by definition with the fundamental units.

Let L, M, T, denote the units of length, mass, and time, and let V, A, F, E, P, denote the derived units of velocity, acceleration, force, energy, and power. Then

$$\text{Velocity} = \frac{\text{length of path described}}{\text{time occupied}}$$

If unit length is described in unit time, the velocity is unity. Hence

$$V = \frac{L}{T}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Likewise, unit acceleration is the acceleration of a body which gains unit velocity in unit time. That is

$$A = \frac{V}{T} = \frac{L}{T^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The absolute unit of force produces unit acceleration in unit mass, and the force is equal to the product of mass and acceleration. Hence

$$F = MA = \frac{ML}{T^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If energy is measured in work units, so that the unit of energy is the work done by unit force in a displacement of unit length in the direction of the force, then

$$E = FL = \frac{ML^2}{T^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If a body of unit mass has unit velocity, its kinetic energy is

$$\tfrac{1}{2}MV^2 = \tfrac{1}{2}M\frac{L^2}{T^2} = \tfrac{1}{2}E \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Thus the kinetic energy of the body is half a unit of energy.

The unit of power is the power of a force which does unit work in unit time. Hence

$$P = \frac{E}{T} = \frac{ML^2}{T^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The preceding units are all absolute units, the most convenient units for theoretical investigations. In practical calculations it is customary to use any convenient unit. The pound is the practical unit of force for example, and in order to make the equation

Force = mass × (acceleration produced)

hold, engineers have invented a new unit of mass which is the mass of g pounds. Thus the engineers' units of force and mass are each g times as large as the theoretical or absolute units. The objection to the pound as a theoretical unit of force is that the weight of one pound depends on the earth's attraction, which is different at different places, but the unit is good enough for practical purposes, and both systems give the same theoretical. results.

The unit of volume is L^3, and the unit of area L^2. Also the unit of density is the density of a body which has unit mass in unit volume. If D denotes this unit,

$$D = \frac{M}{L^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The unit of momentum is

$$MV = \frac{ML}{T} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The impulse of a force f acting for a time t is ft. Hence, if I denotes the unit of impulse,

$$I = FT = \frac{ML}{T^2} \times T = \frac{ML}{T} \quad . \quad . \quad . \quad . \quad (9)$$

which is, of course, the same as the unit of momentum, because an impulse is equal to the momentum generated.

The number of radians in an angle is

$$\theta = \frac{\text{length of arc subtended}}{\text{radius of circle}}$$

If both radius and arc have unit length, we get the unit angle one radian. Calling this R, we get

$$R = \frac{L}{L} = 1, \text{ a number} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Angular velocity is the number of radians described in unit time. Hence, if Ω denote the unit,

$$\Omega = \frac{R}{T} = \frac{1}{T} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

485. Every physical quantity is of the form

$$NL^{\alpha}M^{\beta}T^{\gamma} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where N, α, β, γ, are numbers, and L, M, T, are the units of length, mass, and time. The quantity in (12) is said to be of dimension α in length, β in mass, and γ in time. Two physical quantities have the same dimensions when they contain the same powers of L, M, and T. Every term in any physical or mechanical equation has the same dimensions, for quantities of different dimensions are not like quantities, and cannot, therefore, be added together. For example, a velocity cannot be added to a displacement, or it would follow that a displacement divided by a velocity could be a mere number, whereas we know that when we divide a distance by a velocity the quotient gives the time required by a body having the velocity to describe the distance; that is, distance divided by velocity can only give time.

The usefulness of the theory of dimensions depends on the important fact that all the terms in an equation must have the same dimensions. It enables us to check our working at any stage. If we find two terms of different dimensions in the same equation, we know that there is some mistake. Moreover, when the data in a particular problem are small in number, we can even predict the form of the result merely by using the theory of dimensions.

We shall now work a few examples.

486. Illustrative Examples.

EXAMPLE 1.—*The velocity of a body falling from rest through a distance* h *depends only on* h *and* g. *To find how these quantities are involved in the velocity* v.

We have to combine h and g so as to get a quantity of the dimensions of a velocity. Now h, being a length, contains the first power of L, the unit of length. That is

$$h \propto L \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Also, g being an acceleration, we have

$$g \propto \frac{L}{T^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Let us assume

$$v \propto h^{\alpha}g^{\beta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

This gives

$$\frac{L}{T} \propto L^{\alpha}\frac{L^{\beta}}{T^{2\beta}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$$\propto \frac{L^{\alpha+\beta}}{T^{2\beta}}$$

Since both sides are of the same dimensions, we get, on comparing indices,

$$\alpha + \beta = 1 \quad . \quad . \quad . \quad . \quad . \quad (17)$$

$$2\beta = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

which give $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$. Hence

$$v \propto h^{\frac{1}{2}} g^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad (19)$$

or

$$v = \mathrm{N}\sqrt{hg} \quad . \quad . \quad . \quad . \quad . \quad (20)$$

N being a number.

We know this result is correct, for actually

$$v = \sqrt{2hg} \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Example 2.—*By the same process we will find how* h *and* g *are involved in the time of falling through a distance* h *from rest.*

Let t denote the time, and assume

$$t \propto h^{\alpha} g^{\beta} \quad . \quad . \quad . \quad . \quad . \quad (22,$$

From this

$$\mathrm{T} \propto \mathrm{L}^{\alpha+\beta}.\mathrm{T}^{-2\beta} \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Hence, making the dimensions the same,

$$\alpha + \beta = 0 \quad . \quad . \quad . \quad . \quad . \quad (24)$$

$$-2\beta = 1 \quad . \quad . \quad . \quad . \quad . \quad (25)$$

These give $\alpha = \frac{1}{2}$, $\beta = -\frac{1}{2}$. Therefore

$$t \propto \sqrt{\frac{h}{g}} \quad . \quad . \quad . \quad . \quad . \quad (26)$$

Example 3.—*The time of oscillation of a simple pendulum can depend only on the three quantities,* g, l, *the length of the pendulum, and* β, *the angular amplitude. To find, as far as possible, how these quantities are involved in the time.*

Since β is itself a number, the theory of dimensions does not help us to discover how this quantity is involved. We have, then, to combine l and g to get a quantity of the dimensions of time. Since l is a length, the method of the last example shows that

$$t \propto \sqrt{\frac{l}{g}} \quad . \quad . \quad . \quad . \quad . \quad (27)$$

That is

$$t = \mathrm{N}\sqrt{\frac{l}{g}} \quad . \quad . \quad . \quad . \quad . \quad (28)$$

where N is a number, which may involve β in any manner whatever. We may, therefore, write

$$t = \sqrt{\frac{l}{g}} \cdot \mathrm{F}(\beta) \quad . \quad . \quad . \quad . \quad . \quad (29)$$

where $\mathrm{F}(\beta)$ denotes some function of β to which the present theory gives us no clue.

The numerical coefficient of $\sqrt{\frac{l}{g}}$ cannot contain l or g, since no combination of these will give a number. Consequently, this numerical coefficient is the same for all pendulums, provided β is the same, however large β may be.

If a number of uniform rods, each fixed by a smooth hinge at one end, are allowed to fall from the horizontal position, the preceding reasoning will show that the time required for a rod of length l to fall into the lowest position is an expression of the form given in (28). Consequently, the time of falling of any rod is proportional to $\sqrt{l}$, since g is the same for every rod.

EXAMPLE 4.—*To find the dimensions of the constant of gravitation.*

If κ denotes the constant; m_1, m_2, the masses of two attracting bodies; F, the attraction between them; and r, their distance apart, then

$$F = \frac{\kappa m_1 m_2}{r^2} \quad \ldots \ldots \quad (30)$$

Putting in the proper dimensions of the known quantities,

$$\frac{ML}{T^2} \propto \frac{\kappa M^2}{L^2} \quad \ldots \ldots \quad (31)$$

whence
$$\kappa \propto \frac{L^3}{MT^2}$$

or
$$\propto L^3M^{-1}T^{-2} \quad \ldots \ldots \quad (32)$$

EXAMPLE 5.—*To find the form of the expression for the period of a satellite about a primary of mass* S *assumed fixed.*

The period obviously does not depend on the mass of the satellite, because, other things being the same, the attraction is proportional to the mass of the satellite, and therefore the acceleration, which is proportional to the attraction divided by the mass of the satellite, does not depend on this mass.

The period depends on S, the constant of gravitation κ, the major semi-axis, a, of the orbit, and possibly on the eccentricity of the orbit. But, since the eccentricity is a mere number, the theory of dimensions does not help to discover whether it is involved in the period or not.

Let us assume, then, that the period is

$$t = N \,.\, S^{\alpha}\kappa^{\beta}a^{\gamma} \quad \ldots \ldots \quad (33)$$

Writing the dimensions of these quantities

$$T \propto M^{\alpha}(L^3M^{-1}T^{-2})^{\beta}L^{\gamma} \quad \ldots \ldots \quad (34)$$

Making the dimensions of both sides the same, we get

$$\alpha = \beta \quad \ldots \ldots \quad (35)$$

$$-2\beta = 1 \quad \ldots \ldots \quad (36)$$

$$\gamma = -3\beta \quad \ldots \ldots \quad (37)$$

Solving these and substituting in (33)

$$t = NS^{-\frac{1}{2}}\kappa^{-\frac{1}{2}}a^{\frac{3}{2}} = N\frac{a^{\frac{3}{2}}}{\sqrt{\kappa S}} \quad \ldots \ldots \quad (38)$$

If we were sure that the eccentricity were not involved in the number N, this last equation would prove that the periods of different planets are proportional to the cubes of their mean distances.

EXAMPLE 6.—*To find the form of the expression for the maximum span of a catenary of uniform strength of given material.*

The maximum span clearly depends on the breaking tension per unit area P, the density (mass per unit volume) of the material of the catenary ρ, and the intensity of gravity g.

If s is the maximum span,

$$s = NP^{\alpha}\rho^{\beta}g^{\gamma} \qquad (39)$$

The dimensions of P, which is a force divided by an area, is

$$\frac{ML}{T^2} \times \frac{1}{L^2} = \frac{M}{LT^2} \qquad (40)$$

Inserting the dimensions of the quantities in (39), we get

$$L \propto \left(\frac{M}{LT^2}\right)^{\alpha}\left(\frac{M}{L^3}\right)^{\beta}\left(\frac{L}{T^2}\right)^{\gamma} \qquad (41)$$

To make the dimensions the same we must have

$$\alpha + \beta = 0 \qquad (42)$$

$$2\alpha + 2\gamma = 0 \qquad (43)$$

$$-\alpha - 3\beta + \gamma = 1 \qquad (44)$$

These give $\alpha = 1$, $\beta = -1$, $\gamma = -1$. Therefore

$$S = N\frac{P}{\rho g} \qquad (45)$$

In this result P is measured in absolute units. If British fundamental units are used, P is in poundals per square foot, and consequently $\frac{P}{g}$ is the breaking tension in pounds per square foot. Writing P_1 for $\frac{P}{g}$ the result can be expressed thus

$$s = N\frac{P_1}{\rho} \qquad (46)$$

The number N is shown in Art. 198 to be π.

EXAMPLE 7.—*To show that the attraction of an infinite uniform plane distribution of matter is independent of the distance from the plane.*

The attraction can only depend on the constant of gravitation κ, the surface density ρ, and the distance from the plane d. Now attraction is force on unit mass, and has therefore the dimensions of acceleration. Also ρ is mass per unit area, that is mass divided by area.

The attraction must be given by

$$F = N\rho^{\alpha}\kappa^{\beta}d^{\gamma} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

The dimensional equation from this is

$$\frac{L}{T^2} \propto \left(\frac{M}{L^2}\right)^{\alpha}\left(\frac{L^3}{MT^2}\right)^{\beta} L^{\gamma} \quad . \quad . \quad . \quad . \quad . \quad (48)$$

To make the dimensions the same, we get

$$2\beta = 2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

$$\alpha - \beta = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

$$-2\alpha + 3\beta + \gamma = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

Whence $\alpha = \beta = 1$, $\gamma = 0$. Therefore

$$F = N\kappa\rho \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

The number N is shown in Art. 243 to be 2π.

487. The *measure* of any quantity of a particular kind is the number of times that quantity contains the unit of its own kind. If the unit be changed the measures of all quantities will be changed in consequence. For example, the measure of the radius of the earth with the mile as unit of length is about 4000; but when the unit is a metre its measure is about 64×10^5.

If N and N′ are the measures of the same quantity when U and U′ are the units, then

$$NU = N'U' \quad . \quad . \quad . \quad . \quad . \quad (53)$$

Whence

$$\frac{N}{N'} = \frac{U'}{U} \quad . \quad . \quad . \quad . \quad . \quad (54)$$

that is, the measures are in the inverse ratio of the units. The larger the unit the smaller the measure.

If M^{α} occurs in any quantity, an alteration of the unit of mass from M to M′ will alter the measure of the quantity in the ratio $\left(\frac{M}{M'}\right)^{\alpha}$. A few examples will illustrate these principles.

EXAMPLE I.—*The acceleration of a train is* f *miles per hour per hour; to express this in foot and second units.*

Let L and T denote the units one foot and one second; L′ and T′ the units one mile and one hour. Then, if a denotes the measure of the acceleration in foot-second units

$$a\frac{L}{T^2} = f\frac{L'}{T'^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

Therefore

$$a = f\,\frac{L'}{L}\left(\frac{T}{T'}\right)^2 = f\left(\frac{\text{one mile}}{\text{one foot}}\right)\left(\frac{\text{one second}}{\text{one hour}}\right)^2$$

$$= f\frac{5280}{3600^2} = \frac{11}{27000}f \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

EXAMPLE 2.—*If* g *and* g′ *denote the measures of the acceleration due to gravity in foot-second units and centimetre-second units respectively, we find*

$$\frac{g'}{g} = \frac{A}{A'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

where A and A′ are the British and metric units of acceleration. Hence, since the unit of time is the same in both,

$$\frac{g'}{g} = \frac{L}{L'} = \frac{\text{one foot}}{\text{one centimetre}} = 30 \cdot 5 \quad . \quad . \quad . \quad . \quad (58)$$

Therefore $g' = 30 \cdot 5 \times 32 \cdot 2 = 982 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$

EXAMPLE 3.—*The absolute unit of work is*

$$M\frac{L^2}{T^2}$$

The metric unit is called an *erg* and the British unit is a foot-poundal. Their ratio is

$$\frac{1 \text{ foot-poundal}}{1 \text{ erg}} = \frac{1 \text{ lb.}}{1 \text{ gram}} \cdot \left(\frac{1 \text{ foot}}{1 \text{ centimetre}}\right)^2 = 454 \times (30 \cdot 5)^2 . \quad (60)$$

the time unit being the same. Hence

$$1 \text{ foot-poundal} = 422 \times 10^3 \text{ ergs} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

$$1 \text{ foot-pound} = 422 \times 32 \cdot 2 \times 10^3 = 136 \times 10^5 \text{ ergs} . \quad (62)$$

488. The rest of the units in the metric system can be expressed in terms of British units when the ratios of the three fundamental units are given. In the metric system the *centimetre*, *gram*, and *second*, are the names of the fundamental units of length, mass, and time. Their values in terms of British units are given in a table at the end of the book. The absolute unit of force is called a *dyne* and, as we have mentioned in the last article, the absolute unit of work or energy is an *erg*.

489. Whenever different powers of the same quantity are added together in a physical equation this quantity must be a number of no dimensions in length, mass, or time. If, for example, e^z or $\cos z$ occur in an equation, z must be a number, because e^z and $\cos z$ involve different powers of z. In the theory of struts, Art. 237, the deflection is

$$y = C \cos\sqrt{\frac{F}{EI}}\,x + D \sin\sqrt{\frac{F}{EI}}\,x$$

The quantity $\sqrt{\frac{F}{EI}}x$ must therefore be a number. This can be easily shown to be true by finding the dimensions of F, E, and I.

Again, the angular displacement of a simple pendulum is

$$\theta = A \sin\left(\sqrt{\frac{g}{l}}t + \beta\right)$$

Consequently $\sqrt{\frac{g}{l}}t$ must be a number, and this is easily seen to be true from the dimensions of the quantities involved.

Likewise $\sin^{-1}z$, when it occurs in an integral, is a mere number and not an angle. It is a coincidence that the number can be obtained from trigonometrical tables. It should be remembered that the radian measure of an angle is a mere number because it is the ratio of an arc of a circle to its radius.

EXAMPLES ON CHAPTER XXIV

1. What are the dimensions of the gravitation constant? Assuming that the value of the constant is $\frac{1}{9{\cdot}4 \times 10^8}$ in absolute foot-pound-second units, find its value in centimetre-gram-second (*c.g.s.*) units.

[The dimensions are $\frac{L^3}{MT^2}$. In *c.g.s.* units its value is $\frac{1}{1{\cdot}504 \times 10^7}$.]

2. The velocity of a body which has been attracted from infinity to distance r from the sun depends on the sun's mass S, the distance r, and the gravitation constant κ. What is the form of the velocity?

[The velocity is proportional to $\sqrt{\kappa S r^{-1}}$.]

3. What are the dimensions of Young's modulus and of the modulus of rigidity?

[Each has the dimensions $\frac{M}{LT^2}$.]

4. The velocity of longitudinal waves in an elastic body depends on Young's modulus E, and the density ρ. What is the form of the velocity?

[The velocity is proportional to $\sqrt{\frac{E}{\rho}}$.]

5. The velocity of waves on the surface of deep water depends on the intensity of gravity g, and the wave length λ. How are the quantities involved in the velocity?

[The velocity is proportional to $\sqrt{\lambda g}$.]

6. The couple required to twist a wire of length l and radius r through one radian depends on l, r, and the rigidity modulus n. Assuming that the couple varies inversely as l, find how all three are involved.

The couple varies as $\frac{nr^4}{l}$.]

7. The greatest length of a cylindrical horizontal beam that can be safely supported in a given way under its own weight depends on the breaking

tension P, the density ρ, the radius r, and the intensity of gravity g. What does the theory of dimensions tell us about the greatest length?

[The theory tells that the greatest length varies as $\left(\frac{P}{\rho g}\right)^{\alpha} r^{\beta}$, with the condition that $\alpha + \beta = 1$.]

8. The mass of water that flows over a triangular notch in t seconds, the vertex of the triangle being lowest and at depth h below the surface of the water, depends upon t, h, g, and the density ρ, as well as the vertical angle of the triangle. Show, by considering dimensions, that the mass is proportional to $\rho(gt^2)^{\alpha}h^{\beta}$, α and β being connected by the equation $\alpha + \beta = 3$.

[Actually $\alpha = \frac{1}{2}$, $\beta = \frac{5}{2}$, and the tangent of half the vertical angle is a factor, but the theory of dimensions cannot tell us so much.]

9. Assume that the period of a planet in its orbit does not depend on the minor axis of the orbit, and show from dimensions how it depends on the sun's mass S, the major semi-axis a, and the constant of gravitation κ.

[The period varies as $\sqrt{\frac{a^3}{\kappa S}}$.]

10. The maximum angular velocity at which a disc can be rotated about its axis so as to be safe from bursting depends on the radius r, the density ρ, and the breaking tension P. Find how this angular velocity involves the quantities.

[The angular velocity varies as $\sqrt{\frac{P}{\rho r^2}}$.]

11. The slowest period of oscillation of a tightly stretched string depends on its tension T, its length l, and its mass m. How do these quantities occur in the period?

[The period varies as $\sqrt{\frac{ml}{T}}$.]

12. Suppose the buckling load, P, of a strut depends only on the length l and on the product EI. Find how these quantities are involved.

[$P \propto \frac{EI}{l^2}$.]

13. If the maximum stress in a rotating disk depends only on the density ρ, the radius r, and the angular velocity ω, show that it is proportional to $\rho r^2\omega^2$.

14. The terminal velocity of a sphere falling in a fluid depends on g, on the radius r, on the density ρ and the viscosity μ of the fluid. This last has dimensions $ML^{-1}T^{-1}$. Show that the square of the velocity is proportional to $rg(\rho^2r^3g\mu^{-2})^n$, where n is some number not determined by dimensions.

1. Definition of a Conic and Deduction of its Polar Equation.—A conic is the locus of a point which moves in one plane so that its distance from a fixed point in that plane bears a constant ratio e to its distance from a fixed straight line in that plane. The curve is called an ellipse, a parabola, or an hyperbola, according as the ratio e is less than, equal to, or greater than unity.

We will first find the polar equation to a conic, the fixed point being the pole.

The fixed point and the fixed straight line are called the focus and the directrix respectively.

In Fig. 213, S is the focus and CQ the directrix. P is any point on the conic. LL′ is the chord through the focus parallel to the directrix and it is called the latus rectum. Its length will be denoted by $2l$.

Now $SP = r = e \,.\, PQ = e(RS + SC)$

$= e \,.\, r \cos\theta + e \,.\, LM$

$= er\cos\theta + SL = er\cos\theta + l \quad (1)$

Hence $\dfrac{l}{r} = 1 - e\cos\theta \quad . \quad (2)$

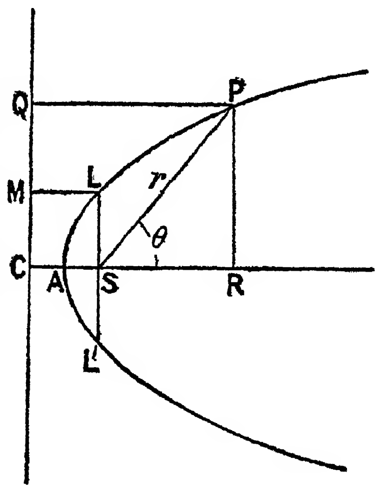


FIG. 213.

If the angle ASP be called θ', we may write the equation (2) in the form

$$\frac{l}{r} = 1 + e\cos\theta' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Either of the equations (2) or (3) may be taken as the polar equation of a conic. In equation (2) the initial line, from which θ is measured, is the aphelion distance, and the initial line for θ' is the perihelion distance.

2. The Ellipse.—Since e is less than unity for the ellipse, it is clear that the curve meets CS produced in some point A′. Let O be the mid-point of AA′.

Then $SA = e \,.\, AC \quad (4)$

$SA' = e \,.\, A'C \quad (5)$

Adding these

$AA' = e(AC + A'C)$

or $2 \,.\, AO = 2e \,.\, CO \quad (6)$

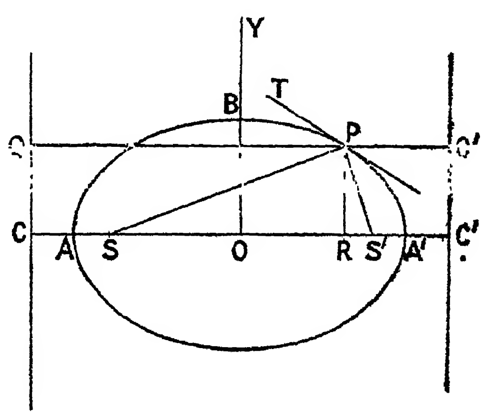


FIG. 214

Writing a for AO, this becomes

$$CO = \frac{a}{e} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Again, subtracting (4) from (5),

$$SA' - SA = e(A'C - AC) = e \cdot AA'$$

or

$$2SO = 2ea$$

that is,

$$SO = ea \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Now taking OA′ and OY as axes of x and y, let the co-ordinates of P be (x, y). Then

$$SP = ePQ = e(CO + OR) = a + ex \quad . \quad . \quad . \quad . \quad (9)$$

Therefore

$$SP^2 = (a + ex)^2$$

But

$$SP^2 = SR^2 + RP^2 = (ea + x)^2 + y^2 . \quad . \quad . \quad . \quad (10)$$

Hence $(ea + x)^2 + y^2 = (a + ex)^2$

Therefore

$$e^2a^2 + 2eax + x^2 + y^2 = a^2 + 2eax + e^2x^2$$

or

$$x^2(1 - e^2) + y^2 = a^2(1 - e^2) . \quad . \quad . \quad . \quad . \quad (11)$$

or

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The value of y when $x = 0$ is $\pm$ OB. Hence putting $x = 0$, we find

$$OB^2 = a^2(1 - e^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Let OB $= b$. Then

$$b^2 = a^2(1 - e^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

and the equation to the ellipse can be written

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

This equation shows that the curve is symmetrical about the axis of y, for there are two equal and opposite values of x for any given value of y. There is consequently another focus S′ and a corresponding directrix at the other side of the curve. If QP meet the other directrix in Q′, we have

$$SP = ePQ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$$S'P = ePQ' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Hence

$$SP + S'P = e \cdot QQ' = 2e \cdot OC = 2a \quad . \quad . \quad (18)$$

That is, the sum of the distances of any point on the ellipse from the foci is constant, and is equal to the major axis AA′.

We shall now prove that SP and S′P make equal angles with the tangent TP.

Let P and Q be two neighbouring points on any curve; and let $SP = r$, $SQ = r + \delta r$, S being any point in the plane of the curve. SF is cut off equal to r so that $FQ = \delta r$. Let the arc PQ be denoted by δs. Then

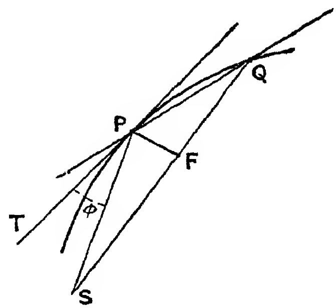


FIG. 215.

$$\frac{dr}{ds} = \lim \frac{FQ}{PQ}$$

$$= \lim \cos PQF$$

$$= \cos \phi$$

By cutting off SF equal to SP we do not make PFQ a right angle, but its limiting value is a right angle, since the triangle SPF is an isosceles triangle which has ultimately its vertical angle zero, and therefore its base angles right angles.

Now in the ellipse, if $S'P = r'$ (Fig. 214),

$$r + r' = 2a, \text{ a constant.}$$

Therefore
$$\frac{dr}{ds} + \frac{dr'}{ds} = 0$$

that is,
$$\cos SPT + \cos S'PT = 0$$

Hence
$$SPT + S'PT = 180^\circ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

and consequently the acute angles between the tangent and the lines SP and S′P are equal.

3. The Auxiliary Circle.—Let a circle be described on AA′ as diameter. This is called the auxiliary circle to an ellipse with AA′ as major axis. If P_1 and P are points on the circle and ellipse respectively such that P_1PR is perpendicular to AA′, then P_1 and P are called corresponding points. The angle $A'OP_1$, denoted by ϕ, is called the eccentric angle of the point P.

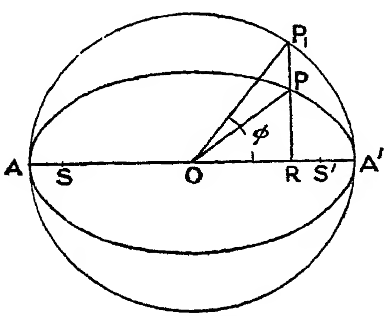


FIG. 216.

If OR be called x and RP, RP_1, be called y and y_1, then

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} \quad . \quad (20)$$

$$\frac{y_1^2}{a^2} = 1 - \frac{x^2}{a^2} \quad . \quad (21)$$

Hence
$$\frac{y}{y_1} = \frac{b}{a} \quad . \quad . \quad . \quad (22)$$

Thus the ratio of the ordinates at corresponding points of the ellipse and circle is constant and equal to $\frac{b}{a}$. This shows that the ellipse could be obtained by turning the circle about AA′ until its plane were inclined to the plane of the ellipse at an angle $\cos^{-1}\frac{b}{a}$ and then projecting the circle orthogonally on the latter plane.

If the circle and the ellipse be divided into very thin strips parallel to the minor axis of the ellipse, it is clear that the areas of the strips of the ellipse would bear the constant ratio $b : a$ to the areas of the corresponding strips of the circle. Consequently the whole area of the ellipse is $\frac{b}{a}$ of the area of the circle, that is, its area is πab.

In terms of the eccentric angle the co-ordinates of P are $a \cos \phi$, $b \sin \phi$.

4. Let FPF′ be a tangent at P; SF and S′F′ perpendiculars to this tangent. We shall prove that F and F′ lie on the auxiliary circle, and that

$$SF . S'F' = b^2 \quad . \quad (23)$$

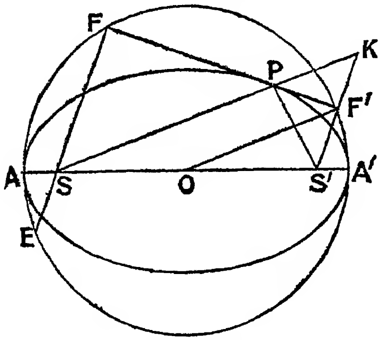


FIG. 217.

Let SP be produced to meet S′F′ in K. Then, since the angles SPF, S′PF′, are equal, it follows that PK = PS′. Hence

$$SK = SP + PS' = 2a \quad (24)$$

Now because O and F′ are the mid-points of S′S and S′K, therefore

$$OF' = \tfrac{1}{2}SK = a \quad (25)$$

Thus F′, and clearly also F, lies on the auxiliary circle.

It is useful to remember also that K, the image of S′ in the tangent, lies on a circle of radius $2a$ with S as centre.

FS is produced to meet the auxiliary circle again in E. Then obviously SE = S′F′. Now let SP = r, S′P = r', SF = p, S′F′ = p'. Then

$$pp' = SF . SE = AS . SA' = a(1-e)a(1+e) = a^2(1-e^2) = b^2 \quad (26)$$

Again, because the angles SPF, S′PF′, are equal

$$\frac{p}{r} = \frac{p'}{r'}$$

Therefore
$$p^2 = p\frac{p'}{r'}r = \frac{b^2}{r'}r = \frac{b^2 r}{2a - r}$$

whence
$$\frac{b^2}{a}\cdot\frac{1}{p^2} = \frac{2}{r} - \frac{1}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

which is the tangential-polar equation to the ellipse.

Now the semi-latus-rectum is the value of y when $x = ea$. Thus equation (15) gives, for the ellipse,

$$\frac{(ae)^2}{a^2} + \frac{l^2}{b^2} = 1$$

whence
$$l^2 = b^2(1 - e^2) = b^2\frac{b^2}{a^2} = \frac{b^4}{a^2}$$

and therefore
$$l = \frac{b^2}{a}$$

Thus equation (27) may be written

$$\frac{l}{p^2} = \frac{2}{r} - \frac{1}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27A)$$

The above is the tangential-polar equation of the ellipse with a focus as pole. We shall now find the tangential-polar equation with the centre as pole.

Let the co-ordinates of P be (x', y'), and let p_1 and r_1 denote respectively the perpendicular distance of O from FF′ and the distance OP. Then it is easy to show that the equation of the tangent is

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=1$$

which becomes, in terms of the eccentric angle ϕ of the point of contact P,

$$\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1$$

The perpendicular p_1 from O on this line is given by

$$\frac{1}{p_1^2}=\frac{\cos^2\phi}{a^2}+\frac{\sin^2\phi}{b^2}$$

whence

$$\begin{aligned}\frac{a^2b^2}{p_1^2}&=b^2\cos^2\phi+a^2\sin^2\phi\\&=a^2+b^2-(a^2\cos^2\phi+b^2\sin^2\phi)\\&=a^2+b^2-(x'^2+y'^2)\\&=a^2+b^2-r_1^2 \quad\quad . \ . \ . \ . \ . \ . \ . \ . \quad (28)\end{aligned}$$

which is the required equation.

5. The Hyperbola.—The polar equation to the hyperbola

$$\frac{l}{r}=1-e\cos\theta$$

shows that r is infinite when

$$\cos\theta=\frac{1}{e} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (29)$$

which gives real values of θ since $e>1$. When $(1-e\cos\theta)$ is positive r is positive, but when this quantity is negative r must be regarded as negative, that is, r must be drawn in the direction opposite to that of the arm bounding the angle θ. By reasoning from the polar equation it can be shown that the curve consists of two infinite branches which do not unite; but it will be easier to see this from the cartesian equation.

By nearly the same working as for the ellipse we can prove that

$$\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1$$

Now $e>1$ for the hyperbola. Let us therefore put b^2 for the positive quantity $a^2(e^2-1)$. Then the equation becomes

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (30)$$

This last equation shows that the least value of x^2 is a^2. Consequently there is no portion of the curve between the lines $x=+a$, $x=-a$.

Where the line

$$y=mx \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (31)$$

meets the hyperbola, we have

$$\frac{x^2}{a^2}-\frac{m^2x^2}{b^2}=1$$

$$\frac{1}{x^2}=\frac{1}{a^2}-\frac{m^2}{b^2} \quad . \ . \ . \ . \ . \ . \ . \ . \quad (32)$$

This gives two real equal and opposite values of x so long as the right-hand side of (32) is positive, that is, so long as

$$m^2 < \frac{b^2}{a^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

When m^2 is equal to $\frac{b^2}{a^2}$ the value of x is infinite, and greater values of m^2 give no real values of x. It follows, then, that there is no portion of the curve in that region between the two lines

$$y = \frac{b}{a}x, \quad y = -\frac{b}{a}x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

which contains the axis of y. These two lines are called asymptotes to the hyperbola. They are the lines OH and OK in the figure.

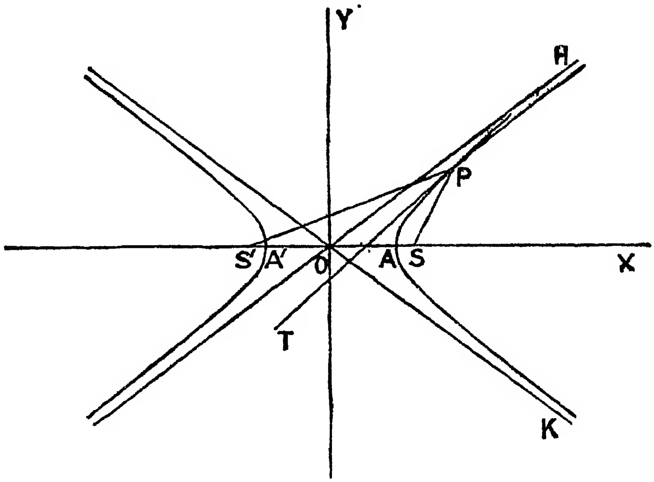


FIG. 218.

If 2α denotes the angle between the asymptotes then

$$\tan^2 \alpha = \frac{b^2}{a^2} = e^2 - 1$$

and therefore $\qquad \sec \alpha = e$

Corresponding to (18) the property of the hyperbola is

$$S'P - SP = AA' = 2a$$

that is, $\qquad r' - r = 2a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$

P being a point on the branch nearest S.

Also, if PT is a tangent to the hyperbola at the point P, we can prove that

$$\text{angle SPT} = \text{angle S'PT} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

Just as for the ellipse we can prove that the feet of the perpendiculars from the foci on a tangent lie on the circle on AA′ as diameter, and thence

$$pp' = b^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

Again, by the method used for the ellipse,

$$p^2 = \frac{pp'}{r'} r = \frac{b^2 r}{2a + r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

Consequently
$$\frac{b^2}{a} \cdot \frac{1}{p^2} = \frac{2}{r} + \frac{1}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

which is the tangential-polar equation to that branch of the hyperbola nearest the pole S. Taking the pole at S′ to get the equation of the further branch

$$p'^2 = \frac{pp'}{r} r' = \frac{b^2 r'}{r' - 2a}$$

whence
$$\frac{b^2}{a} \cdot \frac{1}{p'^2} = \frac{1}{a} - \frac{2}{r'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

As in the ellipse the focus is at (ea, l). Hence equation (30) gives

$$\frac{e^2 a^2}{a^2} - \frac{l^2}{b^2} = 1$$

whence
$$l^2 = (e^2 - 1) b^2 = \frac{b^2}{a^2} b^2$$

and therefore
$$l = \frac{b^2}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

as in the ellipse.

6. The Parabola.—For this curve $e = 1$, and therefore

$$SP = PM.$$

If any line DM′ be taken parallel to the directrix, we find that, when P is between CM and DM′,

$$SP + PM' = MP + PM' = \text{constant}$$

Now imagine DM′ to be moved to a very great distance to the right, so that PD makes an infinitesimal angle with SD. Then PD would differ very little from PM′, and we should have, approximately,

$$SP + PD = \text{a constant} \quad (42)$$

But if SP + PD were exactly constant, the curve would be an ellipse with foci at S and D, and it would follow that

$$\text{angle } SPT = \text{angle } DPT' \quad (43)$$

In the limit when DM′ is at an infinite distance to the right of CM, the relations (42) and (43) will be exactly true. A parabola may therefore be regarded as an ellipse with one focus at infinity, and consequently the centre at infinity. The line joining any given point to the focus at infinity, or to the centre, is a line through

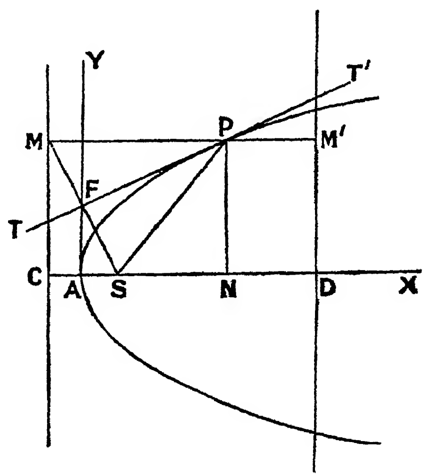


FIG. 219.

that point parallel to the axis AD of the parabola. Thus PM′ is along the second focal distance of P.

Most properties of the parabola can therefore be derived from those of the ellipse by assuming one focus to be at infinity. Thus

$$\text{angle SPT} = \text{angle M'PT'} = \text{angle MPT} \quad . \quad . \quad . \quad (44)$$

Again, since $\frac{b^2}{a}$ is equal to l, the semi-latus-rectum of an ellipse, which remains finite in the parabola, it follows that b becomes infinite when a becomes infinite. From the tangential polar equation to the ellipse

$$\frac{b^2}{a}\cdot\frac{1}{p^2} = \frac{2}{r} - \frac{1}{a}$$

we get that of the parabola thus

$$\frac{l}{p^2} = \frac{2}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

This can be proved directly from (44). Thus join SM. Then in the triangles MPF, SPF, MP and PF are respectively equal to SP and PF, and the included angles are equal. Therefore SM is perpendicular to the tangent, and

$$p = \text{SF} = \text{FM}$$

Therefore $$\frac{p}{r} = \frac{\text{SF}}{\text{SP}} = \frac{\text{CS}}{\text{SM}} \text{ from similar triangles}$$

But CS is equal to l by the property of the curve, and SM $= 2p$. Hence

$$\frac{p}{r} = \frac{l}{2p}$$

which agrees with (45).

The auxiliary circle of the ellipse, which passes through A and has its centre at the centre of the ellipse, becomes a circle of infinite radius in the parabola, that is, a straight line touching the parabola at A. The feet of the perpendiculars from S on the tangents lie on this straight line.

The parabola may be derived not only from the ellipse but also from the hyperbola. If DM′ is on the left of CM, we get

$$\text{PM}' - \text{SP} = \text{a constant}$$

Now, supposing DM′ to be moved to infinity on the left, PM′ is the same as PD, and we find

$$\text{PD} - \text{SP} = \text{a constant}$$

which is the same as for an hyperbola with its second focus D at an infinite distance to the left of S.

We will now find the cartesian equation referred to axes through A, the axis of Y being parallel to the directrix. A is, of course, the mid-point of CS.

By definition

$$\text{SP} = \text{MP} = \text{CA} + \text{AN}$$

Therefore $$\text{SP}^2 = (\text{CA} + \text{AN})^2$$

or $$\text{SN}^2 + \text{NP}^2 = (\text{CA} + \text{AN})^2$$

Now, writing x, y, l, for AN, NP, CS, we get

$$\left(x - \frac{l}{2}\right)^2 + y^2 = \left(\frac{l}{2} + x\right)^2$$

whence $$y^2 = 2lx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

USEFUL QUANTITIES

1 foot	= 30·48 centimetres.
1 inch	= 2·540 centimetres.
1 mile	= 1·6093 kilometres.
1 pound	= 453·59 grams.
1 metre	= 39·37 inches.
1 kilogram	= 2·2046 pounds.
1 poundal	= 13,825 dynes.
1 foot-poundal	= 4217×10^2 ergs.

$\pi = 3·141593$.

$e = 2·718282$ (the base of natural logarithms).

$\log_{10} e = 0·4342945$.

$\log_e 10 = 2·302585$.

$$\left.\begin{aligned} g &= 32·173 - 0·083 \cos 2\theta \\ G &= 32·228 - 0·028 \cos 2\theta \end{aligned}\right\} \text{where } \theta \text{ denotes the latitude.}$$

G is the true acceleration produced by the attraction of the Earth at sea-level, whereas g is the acceleration relative to an observer on the rotating earth. The difference between g and G is the acceleration of the observer due to the earth's rotation on its axis.

At Greenwich $g = 32·191$.

At Paris $g = 32·183$.

One cubic foot of water at 4° C. weighs 62·3 lbs.

12·41 cubic feet of air at 0° C. weigh one lb. when the mercury barometer stands at 30 inches.

Equatorial radius of Earth = 3962·8 miles.

Polar radius of Earth = 3949·5 miles.

Mean distance of Earth from Sun = 92·8 million miles.

Mean distance of Moon from Earth = 60·26 equatorial radii of Earth.

One sidereal month = 27·322 days.

One tropical year = 365·24 days = 13·368 sidereal months.

Mass of Sun = 328,000 times mass of Earth and Moon.

Mass of Earth = 80·4 times mass of Moon.

Semi-diameter of Sun subtends 961″·82 at the Earth's mean distance.

Semi-diameter of Moon subtends 934″·68 at the Earth's mean distance.

The constant of gravitation = $\dfrac{1}{948 \times 10^6}$, the units of mass, time, and force being the pound, second, and poundal, and the mean density of the Earth being taken as 5·6 times that of water.

Coefficients of Friction.

Wood on wood, dry	0·25 to 0·50.
Hemp on oak, dry	0·53.
Hemp on oak, wet	0·33.
Metals on oak, dry	0·50 to 0·60.
Metals on oak, wet	0·24 to 0·26.
Metals on elm, dry	0·20 to 0·25.
Leather on oak	0·27 to 0·35.
Leather on metals, dry	0·56.
Leather on metals, wet	0·36.
Leather on metals, greasy	0·23.
Leather on metals, oiled	0·15.
Metals on metals, dry	0·15 to 0·20.
Metals on metals, wet	0·30.

The following table gives some useful properties of common materials. The density is in pounds per cubic foot, the two moduluses in millions of pounds per square inch, and the breaking tension in thousands of pounds per square inch.

Material.	Density.	Young's modulus.	Modulus of rigidity.	Breaking tension.
Wrought iron bars	480	29	11·1	60 to 70
Cast iron	444	17	2·9	16·5
Steel bars	490	29 to 42	11·8	100 to 120
Copper wire	545	17	7·1	60
Oak	50	1·5	0·082	12
Ash	47	1·6	0·076	17

The following table gives the periods of the planets in days, their mean distances from the Sun in terms of the Earth's mean distance, and the ratio of the mass of the sun (S) to that of the planet (M).

Planet.	Period.	Mean distance.	S ÷ M.
Mercury	87·97	0·3871	8,000,000 (?)
Venus	224·7	0·7233	402,000
Earth	365·24	1	332,000
Mars	687·0	1·524	3,093,000
Jupiter	4,333	5·203	1,048
Saturn	10,759	9·539	3,502
Uranus	30,687	19·18	22,500
Neptune	60,127	29·92	19,000

INDEX

(The numbers refer to the pages)